# Stability of travelling waves

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**Abstract**

An overview of various aspects related to the spectral and nonlinear stability of travelling-wave solutions to partial differential equations is given. The point and the essential spectrum of the linearization about a travelling wave are discussed as is the relation between these spectra, Fredholm properties, and the existence of exponential dichotomies (or Green's functions) for the linear operator. Among the other topics reviewed in this survey are the nonlinear stability of waves, the stability and interaction of well-separated multi-bump pulses, the numerical computation of spectra, and the Evans function, which is a tool to locate isolated eigenvalues in the point spectrum and near the essential spectrum. Furthermore, methods for the stability of waves in Hamiltonian and monotone equations as well as for singularly perturbed problems are mentioned. Modulated waves, rotating waves on the plane, and travelling waves on cylindrical domains are also discussed briefly.

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# Contents

# 1 Introduction

This survey is devoted to the stability of travelling waves. Travelling waves are solutions to partial differential equations that move with constant speed $c$ while maintaining their shape. In other words, if the solution is written as $U(x,t)$ where $x$ and $t$ denote the spatial and time variable, respectively, then we have $U(x,t) = Q(x-ct)$ for some appropriate function $Q(\xi)$. Note that $c = 0$ describes standing waves that do not move at all. In homogeneous media, travelling waves arise as one-parameter families: any translate $Q(x+\tau-ct)$ of the wave $Q(x-ct)$, with $\tau \in \mathbb{R}$ fixed, is also a travelling wave.

We can distinguish between various different shapes of travelling waves (see Figure 1): Wave trains are spatially-periodic travelling waves so that $Q(\xi+L) = Q(\xi)$ for all $\xi$ for some $L > 0$. Homogeneous waves are steady states that do not depend on $\xi$ so that $Q(\xi) = Q_0$ for all $\xi$. Fronts, backs and pulses are travelling waves that are asymptotically constant, i.e. that converge to homogeneous rest states: $\lim_{\xi\to\pm\infty} Q(\xi) = Q_\pm$. For fronts and backs, we have $Q_+ \neq Q_-$, whereas pulses converge towards the same rest state as $\xi \to \pm\infty$ so that $Q_+ = Q_-$.

Travelling waves arise in many applied problems. Such waves play an important role in mathematical biology (see e.g. [121]) where they describe, for instance, the propagation of impulses in nerve fibers. Various different kinds of waves can often be observed in chemical reactions [99, 182]; one example are flame fronts that arise in problems in combustion [182]. Another field where waves arise prominently is nonlinear optics (see e.g. [1]): of interest there are models for the transmission and propagation of beams and pulses through optical fibers or waveguides. We refer to [38, 43, 89] for applications to water waves. Travelling waves also arise as viscous shock profiles in conservation laws that model, for instance, problems in fluid and gas dynamics or magneto-hydrodynamics [171]. Localized structures in solid mechanics can be modelled by standing waves (see [172, 173, 174]). We refer to [59] for the existence and stability of patterns on bounded domains.

In this article, we focus on the stability of a given travelling wave. That is, we are interested in the fate of solutions whose initial conditions are small perturbations of the travelling wave under consideration. If any such solution stays close to the set of all translates of the travelling wave $Q(\cdot)$ for all positive times, then we say that the travelling wave $Q(\cdot)$ is stable. If there are initial conditions arbitrarily close to the wave such that the associated solutions leave a small neighbourhood of the wave and its translates, then the wave is said to be unstable. In other words, we are interested in orbital stability of travelling waves.

There exists an enormous number of different approaches to investigate the stability of waves: which of these is the most appropriate depends, for instance, on whether the partial differential equation is dissipative or conservative, or whether one can exploit a special structure such as monotonicity or singular perturbations. Given this variety, writing a comprehensive survey is quite difficult: thus, the selection of topics appearing in this survey is a very personal one and, of course, by no means complete. We refer to the recent review [184] and to the monograph [182] for many other references related to the existence and stability of waves.

A natural approach to the study of stability of a given travelling wave $Q$ is to linearize the partial differential equation about the wave. The spectrum of the resulting linear operator $\mathcal{L}$ should then provide clues as to the stability of the wave with respect to the full nonlinear equation. As we shall see in Section 3, the spectrum

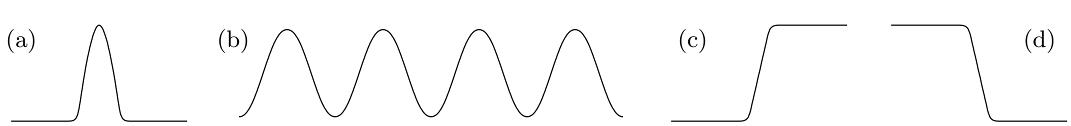


**Figure 1** Travelling waves with various different shapes are plotted: pulses in (a), spatially-periodic wave trains in (b), fronts in (c), and backs in (d). Note that the distinction between fronts and backs is, in general, rather artificial.

of $\mathcal{L}$ is the union of the point spectrum, defined as the set of isolated eigenvalues with finite multiplicity, and its complement, the essential spectrum. Point and essential spectra are also related to Fredholm properties of the operator $\mathcal{L} - \lambda$. Most of the results presented here are formulated using the first-order operator $\mathcal{T}(\lambda)$ that is obtained by casting the eigenvalue operator $\mathcal{L} - \lambda$ as a first-order differential operator. In Section 4, we review the definition and properties of the Evans function, which is a tool to locate and track the point spectrum of $\mathcal{L}$. In Section 7, we discuss under what conditions spectral stability of the linearization $\mathcal{L}$ implies nonlinear stability, i.e. stability of the wave with respect to the full partial differential equation. The stability analysis of a given wave is often facilitated by exploiting the structure of the underlying equation. In Section 8, we provide some pointers to the literature for Hamiltonian and monotone equations as well as for singularly perturbed problems. In many applications, it appears to be difficult to analyze the stability of travelling waves analytically. For this reason, we comment in Section 6 on the numerical computation of the spectra of linearizations about travelling waves. An interesting problem that is relevant for a number of applications is the stability of multi-bump pulses that accompany primary pulses. Recent results in this direction are reviewed in Section 5. Most of the results presented in this survey are also applicable to other waves, for instance, rotating waves such as spiral waves in two space dimensions, modulated waves (waves that are time-periodic in an appropriate moving frame), and travelling waves on cylindrical domains. Some of these extensions are discussed in Section 9.

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# 2 Set-up and examples

## 2.1 Set-up

We consider partial differential equations (PDEs) of the form

$$U_t = \mathcal{A}(\partial_x)U + \mathcal{N}(U), \qquad x \in \mathbb{R}, \qquad U \in \mathcal{X}. \tag{2.1}$$

Here, $\mathcal{A}(z)$ is a vector-valued polynomial in $z$, and $\mathcal{X}$ is an appropriate Banach space consisting of functions $U(x)$ with $x \in \mathbb{R}$, so that $\mathcal{A}(\partial_x) : \mathcal{X} \to \mathcal{X}$ is a closed, densely defined operator. Lastly, $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ denotes a nonlinearity, perhaps not defined on the entire space $\mathcal{X}$, that is defined via pointwise evaluation of $U$ and, possibly, derivatives of $U$. We refer to [85, 133] for more background.

Travelling waves are solutions to (2.1) of the form $U(x,t) = Q(x-ct)$. Introducing the coordinate $\xi = x-ct$, we seek functions $U(\xi,t) = U(x-ct,t)$ that satisfy (2.1). In the $(\xi,t)$-coordinates, the PDE (2.1) reads

$$U_t = \mathcal{A}(\partial_\xi)U + c\partial_\xi U + \mathcal{N}(U), \qquad \xi \in \mathbb{R}, \qquad U \in \mathcal{X}, \tag{2.2}$$

and the travelling wave is then a stationary solution $Q(\xi)$ that satisfies

$$0 = \mathcal{A}(\partial_\xi)U + c\partial_\xi U + \mathcal{N}(U). \tag{2.3}$$

The linearization of (2.2) about the steady state $Q(\xi)$ is given by

$$U_t = \mathcal{A}(\partial_\xi)U + c\partial_\xi U + \partial_U\mathcal{N}(Q)U. \tag{2.4}$$

The right-hand side defines the linear operator

$$\mathcal{L} := \mathcal{A}(\partial_\xi) + c\partial_\xi + \partial_U\mathcal{N}(Q).$$

Spectral stability of the wave $Q$ is determined by the spectrum of the operator $\mathcal{L}$, i.e. by the eigenvalue problem

$$\lambda U = \mathcal{A}(\partial_\xi)U + c\partial_\xi U + \partial_U \mathcal{N}(Q)U = \mathcal{L}U \tag{2.5}$$

that determines whether (2.4) supports solutions of the form $U(\xi, t) = \mathrm{e}^{\lambda t}U(\xi)$.

Note that the steady-state equation (2.3) and the eigenvalue problem (2.5) are both ordinary differential equations (ODEs). As such, they can be cast as first-order systems. The steady-state equation, for instance, can be written as

$$u' = f(u, c), \qquad u \in \mathbb{R}^n, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}\xi}. \tag{2.6}$$

The travelling wave $Q(\xi)$ corresponds to a bounded solution $q(\xi)$ of (2.6). The PDE eigenvalue problem (2.5) becomes

$$u' = (\partial_u f(q(\xi), c) + \lambda B)u, \tag{2.7}$$

where $B$ is an appropriate $n \times n$ matrix that encodes the PDE structure (see Section 2.2 below for examples). An important relation is given by

$$\partial_c f(u(\xi), c) = -Bu'(\xi) \tag{2.8}$$

which follows from inspecting (2.5).

In this survey, we focus on the ODE formulation (2.6) and (2.7). In particular, travelling waves can be sought as bounded solutions of (2.6), and we refer to the textbooks [33, 80, 107] for a dynamical-systems approach to constructing such solutions.

## 2.2 Examples

We give a few examples that fit into the framework outlined above.

**Example 1 (Reaction-diffusion systems)** Let $D$ be a diagonal $N \times N$ matrix with positive entries and $F : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth function. Consider the reaction-diffusion equation

$$U_t = DU_{xx} + F(U), \qquad x \in \mathbb{R}, \qquad U \in \mathbb{R}^N, \tag{2.9}$$

posed on the space $\mathcal{X} = C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{R}^N)$ of bounded, uniformly continuous functions. In the moving frame $\xi = x - ct$, the system (2.9) is given by

$$U_t = DU_{\xi\xi} + cU_\xi + F(U), \qquad \xi \in \mathbb{R}, \qquad U \in \mathbb{R}^N. \tag{2.10}$$

Suppose that $U(\xi, t) = Q(\xi)$ is a stationary solution of (2.10) such that

$$DQ_{\xi\xi}(\xi) + cQ_\xi(\xi) + F(Q(\xi)) = 0, \qquad \xi \in \mathbb{R}. \tag{2.11}$$

The eigenvalue problem associated with the linearization of (2.10) about $Q(\xi)$ is given by

$$\lambda U = DU_{\xi\xi} + cU_\xi + \partial_U F(Q)U =: \mathcal{L}U. \tag{2.12}$$

This eigenvalue problem can be cast as

$$\begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} V \\ D^{-1}(\lambda U - \partial_U F(Q(\xi))U - cV) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\lambda - \partial_U F(Q(\xi))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

which we write as

$$u_\xi = A(\xi; \lambda)u = (\tilde{A}(\xi) + \lambda B)u, \qquad u \in \mathbb{R}^n = \mathbb{R}^{2N} \tag{2.13}$$

with $u = (U, V)$ and

$$\tilde{A}(\xi) = \begin{pmatrix} 0 & \mathrm{id} \\ -D^{-1}\partial_U F(Q(\xi)) & -cD^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ D^{-1} & 0 \end{pmatrix}.$$

Bounded solutions to (2.12), namely $(\mathcal{L} - \lambda)U = 0$, and (2.13) are then in one-to-one correspondence.

In particular, if $Q(\cdot)$ is not a constant function, then $\lambda = 0$ is an eigenvalue of $\mathcal{L}$ with eigenfunction $Q_\xi(\xi)$. This can be seen by taking the derivative of (2.11) with respect to $\xi$ which gives

$$D(Q_\xi)_{\xi\xi} + c(Q_\xi)_\xi + \partial_U F(Q)Q_\xi = 0$$

so that $\mathcal{L}Q_\xi = 0$. Hence, $u(\xi) = (Q_\xi(\xi), Q_{\xi\xi}(\xi))$ satisfies (2.13) for $\lambda = 0$.

One important example is the FitzHugh–Nagumo equation (FHN)

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \delta^2 w_{xx} + \epsilon(u - \gamma w), \end{aligned}$$

for instance with $f(u) = u(1-u)(u-a)$. It admits various travelling waves such as pulses, fronts and backs (see e.g. [91, 105, 176] for references). The stability of pulses has been studied in [90, 185]. Stability results for spatially-periodic wave trains can be found in [53, 156], whereas the stability of concatenated fronts and backs has been studied in [124, 147] and [125, 154].

Many other results on the stability of waves to reaction-diffusion equations can be found in the literature (see e.g. [47, 60]). One class of such equations that has been studied extensively are monotone systems (see [37, 141, 182] and Section 8). We also refer to [85, Section 5.4] for instructive examples. ∎

**Example 2 (Phase-sensitive amplification)** The dissipative fourth-order equation

$$U_t + (\partial_{\xi\xi} + 2U^2 - (2\kappa - \eta^2))(\partial_{\xi\xi} + 2U^2 - \eta^2)U + 4\sigma(3U(\partial_\xi U)^2 + U^2\partial_{\xi\xi}U) = 0 \tag{2.14}$$

models the transmission of pulses in optical storage loops under phase-sensitive amplification (see [106]). This equation with $\sigma = 0$ admits the explicit solution

$$Q(\xi) = \eta\,\mathrm{sech}(\eta\xi)$$

for every $\kappa \geq 0$. Its stability has been investigated in [3, 106]. For $\sigma > 0$, (2.14) has multi-bump pulses whose existence and stability has been analyzed in [150]. ∎

**Example 3 (Korteweg–de Vries equation)** The generalized Korteweg–de Vries equation (KdV) is given by

$$U_t + U_{xxx} + U^p U_x = 0, \qquad x \in \mathbb{R},$$

where $p$ is a positive parameter. Formulated in the moving frame $\xi = x - ct$, the generalized Korteweg–de Vries equation reads

$$U_t + U_{\xi\xi\xi} - cU_\xi + U^p U_\xi = 0, \qquad \xi \in \mathbb{R},$$

where $c$ denotes the wave speed. This equation admits a family of pulses given by

$$Q(\xi) = \left[\frac{c(p+1)(p+2)}{2}\right]^{\frac{1}{p}} \mathrm{sech}^{\frac{2}{p}}\left(\frac{p\sqrt{c}\xi}{2}\right)$$

for any positive values of $c$ and $p$. The stability of these solitons was investigated in [10], whereas asymptotic stability has been studied in [134, 135]. The KdV equation is Hamiltonian for all $p > 0$ and known to be completely integrable for $p = 1, 2$. ∎

**Example 4 (Nonlinear Schrödinger equation)** The nonlinear Schrödinger equation (NLS) reads

$$\mathrm{i}\Phi_t + \Phi_{\xi\xi} + 4|\Phi|^2\Phi = 0, \qquad \xi \in \mathbb{R},$$

with $\Phi \in \mathbb{C}$. If we seek solutions of the form $\Phi(\xi, t) = \mathrm{e}^{\mathrm{i}\omega t}U(\xi, t)$, then we obtain the equation

$$\mathrm{i}U_t + U_{\xi\xi} - \omega U + 4|U|^2U = 0, \qquad \xi \in \mathbb{R}, \tag{2.15}$$

where $U \in \mathbb{C}$ and $\omega > 0$. It is known [183] to support stable pulses given by

$$Q(\xi) = \sqrt{\frac{\omega}{2}}\,\mathrm{sech}(\sqrt{\omega}\,\xi).$$

The NLS equation is Hamiltonian and, in fact, completely integrable. Of interest is the persistence and stability of these waves upon adding perturbations that model various physical imperfections. An important example is the perturbation to the dissipative complex cubic-quintic Ginzburg–Landau equation (CGL)

$$\mathrm{i}U_t + U_{\xi\xi} - \omega U + 4|U|^2U + 3\alpha|U|^4U = \mathrm{i}\epsilon(d_1U_{\xi\xi} + d_2U + d_3|U|^2U + d_4|U|^4U) \tag{2.16}$$

for small $\alpha \in \mathbb{R}$ and $\epsilon > 0$. We refer to [177, 97, 98, 1] for various stability and instability results for solitary waves to this equation. Mathematically, the transition from (2.15) to (2.16) is interesting since the perturbation destroys the Hamiltonian nature of (2.15). ■

We refer to [99, 121, 171, 176, 182] for problems where reaction-diffusion equations arise naturally. Formal derivations of the KdV, NLS and CGL equation can be found in [1] and [38, 89] for problems in nonlinear optics and for water waves, respectively.

# 3 Spectral stability

In this section, we review results on the structure of the spectrum of the linearization of a nonlinear PDE about a travelling wave.

**Notation** Throughout this survey, we denote the range and the null space of an operator $\mathcal{L}$ by $\mathrm{R}(\mathcal{L})$ and $\mathrm{N}(\mathcal{L})$, respectively. Eigenvalues of operators and matrices are always counted with algebraic multiplicity.

Consider a matrix $A \in \mathbb{C}^{n\times n}$. We often refer to the eigenvalues of the matrix $A$ as the spatial eigenvalues. We say that $A$ is hyperbolic if all eigenvalues of $A$ have non-zero real part, i.e. if $\mathrm{spec}(A) \cap \mathrm{i}\mathbb{R} = \emptyset$. We refer to eigenvalues of $A$ with positive (negative) real part as unstable (stable) eigenvalues. Similarly, the generalized eigenspace of $A$ associated with all eigenvalues with positive (negative) real part is called the unstable (stable) eigenspace of $A$.

The $\delta$-neighbourhood of an element or a subset $z$ of a vector space is denoted by $\mathcal{U}_\delta(z)$.

## 3.1 Reformulation

As mentioned above, it is often advantageous to write the eigenvalue problem associated with the linearization as a first-order ODE. Therefore, we consider the family $\mathcal{T}$ of linear operators defined by

$$\mathcal{T}(\lambda)\ :\qquad \mathcal{D} \longrightarrow \mathcal{H}, \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u$$

for $\lambda \in \mathbb{C}$. We take either

$$\mathcal{D} = C^1_{\mathrm{unif}}(\mathbb{R}, \mathbb{C}^n), \qquad \mathcal{H} = C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{C}^n)$$

or

$$\mathcal{D} = H^1(\mathbb{R}, \mathbb{C}^n), \qquad \mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n). \tag{3.1}$$

Throughout this survey, we assume that the following hypothesis is met.

**Hypothesis 3.1** *The matrix-valued function $A(\xi;\lambda) \in \mathbb{C}^{n\times n}$ is of the form*

$$A(\xi;\lambda) = \tilde{A}(\xi) + \lambda B(\xi)$$

*where $\tilde{A}(\cdot)$ and $B(\cdot)$ are in $C^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$.*

The operators $\mathcal{T}(\lambda)$ are closed, densely defined operators in $\mathcal{H}$ with domain $\mathcal{D}$. We are interested in the set of $\lambda$ for which $\mathcal{T}(\lambda)$ is not invertible.

## 3.2 Exponential dichotomies

Spectral properties of $\mathcal{T}$ can be classified by using properties of the associated ODE

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi;\lambda)u \tag{3.2}$$

with $u \in \mathbb{C}^n$. We denote by $\Phi(\xi,\zeta)$ the evolution operator[1] associated with (3.2). Note that $\Phi(\xi,\zeta) = \Phi(\xi,\zeta;\lambda)$ depends on $\lambda$, but we often suppress this dependence in our notation.

A particularly useful notion associated with linear ODEs such as (3.2) is exponential dichotomies. Suppose that we consider a linear constant-coefficient equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\lambda)u, \tag{3.3}$$

so that $A(\lambda)$ does not depend on $\xi$. We want to classify solutions to (3.3) according to their asymptotic behaviour as $|\xi| \to \infty$. Suppose that the matrix $A(\lambda)$ is hyperbolic so that the spatial spectrum $\mathrm{spec}(A(\lambda))$ has no points on the imaginary axis. Consequently,

$$\mathbb{C}^n = E_0^{\mathrm{s}}(\lambda) \oplus E_0^{\mathrm{u}}(\lambda) \tag{3.4}$$

where the two spaces on the right-hand side are the generalized stable and unstable eigenspaces of the matrix $A(\lambda)$. We denote by $P_0^{\mathrm{s}}(\lambda)$ the spectral projection of $A(\lambda)$, so that

$$\mathrm{R}(P_0^{\mathrm{s}}(\lambda)) = E_0^{\mathrm{s}}(\lambda), \qquad \mathrm{N}(P_0^{\mathrm{s}}(\lambda)) = E_0^{\mathrm{u}}(\lambda). \tag{3.5}$$

These subspaces are invariant under the evolution $\Phi(\xi,\zeta) = \mathrm{e}^{A(\lambda)(\xi-\zeta)}$ of (3.3). Furthermore, solutions $u(\xi)$ with initial conditions $u(\zeta)$ in $E_0^{\mathrm{s}}(\lambda)$ decay exponentially for $\xi > \zeta$, while solutions with initial conditions $u(\zeta)$ in $E_0^{\mathrm{u}}(\lambda)$ decay exponentially for $\xi < \zeta$. We are interested in a similar characterization of solutions to the more general equation (3.2):

**Definition 3.1 (Exponential dichotomies)** *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that (3.2), with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on $I$ if constants $K > 0$ and $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ exist as well as a family of projections $P(\xi)$, defined and continuous for $\xi \in I$, such that the following is true for $\xi, \zeta \in I$.*

- *With $\Phi^{\mathrm{s}}(\xi,\zeta) := \Phi(\xi,\zeta)P(\zeta)$, we have*

$$|\Phi^{\mathrm{s}}(\xi,\zeta)| \le K\mathrm{e}^{\kappa^{\mathrm{s}}(\xi-\zeta)}, \qquad \xi \ge \zeta, \qquad \xi,\zeta \in I.$$

[1] i.e. $\Phi(\xi,\xi) = \mathrm{id}$, $\Phi(\xi,\tau)\Phi(\tau,\zeta) = \Phi(\xi,\zeta)$ for all $\xi,\tau,\zeta \in \mathbb{R}$ and $u(\xi) = \Phi(\xi,\zeta)u_0$ satisfies (3.2) for every $u_0 \in \mathbb{C}^n$

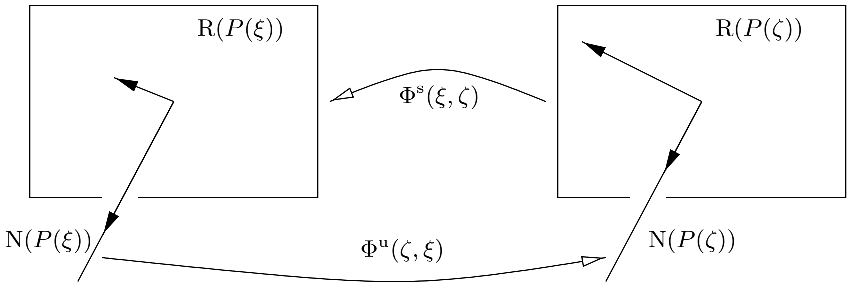


**Figure 2** A plot of the stable and unstable spaces associated with an exponential dichotomy. Vectors in the stable space $\mathrm{R}(P(\zeta))$ are contracted exponentially under the linear evolution $\Phi^{\mathrm{s}}(\xi,\zeta)$ for $\xi > \zeta$. Similarly, vectors in the unstable space $\mathrm{N}(P(\xi))$ are contracted under the linear evolution $\Phi^{\mathrm{u}}(\zeta,\xi)$ for $\xi > \zeta$.

- *Define* $\Phi^{\mathrm{u}}(\xi,\zeta) := \Phi(\xi,\zeta)(\mathrm{id} - P(\zeta))$*, then*

$$|\Phi^{\mathrm{u}}(\xi,\zeta)| \le K \mathrm{e}^{\kappa^{\mathrm{u}}(\xi-\zeta)}, \qquad \xi \le \zeta, \qquad \xi,\zeta \in I.$$

- *The projections commute with the evolution,* $\Phi(\xi,\zeta)P(\zeta) = P(\xi)\Phi(\xi,\zeta)$*, so that*

$$\begin{aligned} \Phi^{\mathrm{s}}(\xi,\zeta)u_0 &\in \mathrm{R}(P(\xi)), \qquad && \xi \ge \zeta, \qquad && \xi,\zeta \in I \\ \Phi^{\mathrm{u}}(\xi,\zeta)u_0 &\in \mathrm{N}(P(\xi)), \qquad && \xi \le \zeta, \qquad && \xi,\zeta \in I. \end{aligned}$$

*The $\xi$-independent dimension of* $\mathrm{N}(P(\xi))$ *is referred to as the Morse index of the exponential dichotomy on $I$. If (3.2) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$, the associated Morse indices are denoted by $i_+(\lambda_*)$ and $i_-(\lambda_*)$, respectively.*

Roughly speaking, (3.2) has an exponential dichotomy on an unbounded interval $I$ if each solution to (3.2) on $I$ decays exponentially either in forward time or else in backward time. The set of initial conditions $u(\zeta)$ leading to solutions $u(\xi)$ that decay for $\xi > \zeta$, with $\xi,\zeta \in I$, is given by the range $\mathrm{R}(P(\zeta))$ of the projection $P(\zeta)$. Similarly, the set of initial conditions $u(\zeta)$ leading to solutions $u(\xi)$ that decay for $\xi < \zeta$, with $\xi,\zeta \in I$, is given by the null space $\mathrm{N}(P(\zeta))$. The spaces $\mathrm{R}(P(\xi))$ are mapped into each other by the evolution associated with (3.2); this is also true for the spaces $\mathrm{N}(P(\xi))$; see Figure 2 for an illustration. For the constant-coefficient equation (3.3), we have $P(\xi) = P_0^{\mathrm{s}}(\lambda)$ due to (3.5). Note that, for constant-coefficient equations, the Morse index of the exponential dichotomy is simply the dimension of the generalized unstable eigenspace.

Exponential dichotomies persist under small perturbations of the equation. This result is often referred to as the roughness theorem for exponential dichotomies.

If we, for instance, perturb the coefficient matrix $A(\lambda)$ of the constant-coefficient equation (3.3) by adding a small $\xi$-dependent matrix, we expect that the two subspaces $E_0^{\mathrm{s}}(\lambda)$ and $E_0^{\mathrm{u}}(\lambda)$ that appear in (3.4) perturb slightly to two new $\xi$-dependent subspaces that contain all initial conditions that lead to exponentially decaying solutions for forward or backward times.

**Theorem 3.1 ([36, Chapter 4])** *Firstly, let $I$ be $\mathbb{R}^+$ or $\mathbb{R}^-$. Suppose that $A(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ and that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi)u \tag{3.6}$$

*has an exponential dichotomy on $I$ with constants $K$, $\kappa^{\mathrm{s}}$ and $\kappa^{\mathrm{u}}$ as in Definition 3.1. There are then positive constants $\delta_*$ and $C$ such that the following is true. If $B(\cdot) \in C^0(I, \mathbb{C}^{n\times n})$ such that $\sup_{\xi\in I, |\xi|\ge L} |B(\xi)| < \delta/C$*

*for some $\delta < \delta_*$ and some $L \geq 0$, then a constant $\tilde{K} > 0$ exists such that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = (A(\xi) + B(\xi))u \tag{3.7}$$

*has an exponential dichotomy on $I$ with constants $\tilde{K}$, $\kappa^{\mathrm{s}} + \delta$ and $\kappa^{\mathrm{u}} - \delta$. Moreover, the projections $P(\xi)$ and evolutions $\Phi^{\mathrm{s}}(\xi,\zeta)$ and $\Phi^{\mathrm{u}}(\xi,\zeta)$ associated with (3.7) are $\delta$-close to those associated with (3.6) for all $\xi,\zeta \in I$ with $|\xi|,|\zeta| \geq L$. Secondly, if $I = \mathbb{R}$, then the above statement is true with $L = 0$.*

Thus, to get persistence of exponential dichotomies on $\mathbb{R}^+$ or $\mathbb{R}^-$, the coefficient matrices of the perturbed equation need to be close to those of the unperturbed equation only for all sufficiently large values of $|\xi|$. For $I = \mathbb{R}$, the coefficient matrices need to be close for all $\xi \in \mathbb{R}$ to get persistence.

Theorem 3.1 can be proved by applying Banach's fixed-point theorem to an appropriate integral equation whose solutions are precisely the evolution operators that appear in Definition 3.1 (see [143, 137]). Indeed, suppose that $\Phi^{\mathrm{s}}(\xi,\zeta)$ and $\Phi^{\mathrm{u}}(\xi,\zeta)$ denote exponential dichotomies of (3.6) on $I = \mathbb{R}^+$, say. If $\sup_{\xi\geq 0}|B(\xi)|$ is sufficiently small, then the dichotomies $\tilde{\Phi}^{\mathrm{s}}(\xi,\zeta)$ and $\tilde{\Phi}^{\mathrm{u}}(\xi,\zeta)$ associated with (3.7) can be found as the unique solution of the integral equation

$$\begin{aligned} 0 &= \tilde{\Phi}^{\mathrm{s}}(\xi,\zeta) - \Phi^{\mathrm{s}}(\xi,\zeta) + \int_\xi^\infty \Phi^{\mathrm{u}}(\xi,\tau)B(\tau)\tilde{\Phi}^{\mathrm{s}}(\tau,\zeta)\,\mathrm{d}\tau \\ &\quad - \int_\zeta^\xi \Phi^{\mathrm{s}}(\xi,\tau)B(\tau)\tilde{\Phi}^{\mathrm{s}}(\tau,\zeta)\,\mathrm{d}\tau + \int_0^\zeta \Phi^{\mathrm{s}}(\xi,\tau)B(\tau)\tilde{\Phi}^{\mathrm{u}}(\tau,\zeta)\,\mathrm{d}\tau, \qquad 0 \leq \zeta \leq \xi \\ 0 &= \tilde{\Phi}^{\mathrm{u}}(\xi,\zeta) - \Phi^{\mathrm{u}}(\xi,\zeta) - \int_\zeta^\xi \Phi^{\mathrm{u}}(\xi,\tau)B(\tau)\tilde{\Phi}^{\mathrm{u}}(\tau,\zeta)\,\mathrm{d}\tau \\ &\quad - \int_0^\xi \Phi^{\mathrm{s}}(\xi,\tau)B(\tau)\tilde{\Phi}^{\mathrm{u}}(\tau,\zeta)\,\mathrm{d}\tau - \int_\zeta^\infty \Phi^{\mathrm{u}}(\xi,\tau)B(\tau)\tilde{\Phi}^{\mathrm{s}}(\tau,\zeta)\,\mathrm{d}\tau, \qquad 0 \leq \xi \leq \zeta \end{aligned} \tag{3.8}$$

(see [143, 137] for details). We emphasize that exponential dichotomies are not unique: on $\mathbb{R}^+$, for instance, the range of $P(\xi)$ is uniquely determined, whereas the null space of $P(0)$ can be chosen to be any complement of $\mathrm{R}(P(0))$; any such choice then determines the null space of $P(\xi)$ for any $\xi > 0$ by the requirement that the projections and the evolution operators commute. The above integral equation fixes such a complement.

**Remark 3.1** *If the perturbation $B(\xi)$ in (3.7) converges to zero as $|\xi| \to \infty$ with $\xi \in I$, then the projections and evolutions of (3.7) converge to those of (3.6) (see e.g. [143, 137]).*

*It is also true that, if (3.2) has an exponential dichotomy for $\lambda = \lambda_*$, then the evolutions and projections that appear in Definition 3.1 can be chosen to depend analytically on $\lambda$ for $\lambda$ close to $\lambda_*$ (see again [143, 137]).*

It is often of interest to distinguish solutions according to the strength of the decay or growth. For instance, for the constant-coefficient system (3.3), we might be interested in distinguishing solutions $u_1(\xi)$ that satisfy

$$|u_1(\xi)| \leq \mathrm{e}^{(\eta-\delta)\xi}|u_1(0)|, \qquad \xi > 0$$

from solutions $u_2(\xi)$ that satisfy

$$|u_2(\xi)| \leq \mathrm{e}^{(\eta+\delta)\xi}|u_2(0)|, \qquad \xi < 0$$

for some chosen $\eta$ and some small $\delta > 0$. In other words, rather than separating stable and unstable eigenvalues of $A(\lambda)$, we divide the spectrum $\mathrm{spec}(A(\lambda))$ into two disjoint sets according to the presence of a spectral gap at $\mathrm{Re}\,\nu = \eta$; see Figure 3b. This may sound more general than the situation considered above, but, in fact, it is not: the scaling

$$v(\xi) = u(\xi)\mathrm{e}^{-\eta\xi} \tag{3.9}$$

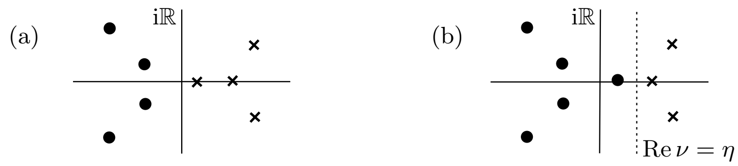


**Figure 3** Two different spectral decompositions of $\text{spec}(A(\lambda))$ are plotted: in (a), stable and unstable eigenvalues are separated by the imaginary axis, whereas the spectrum in (b) is divided into the two disjoint sets $\{\nu \in \text{spec}(A(\lambda));\, \text{Re}\,\nu < \eta\}$ and $\{\nu \in \text{spec}(A(\lambda));\, \text{Re}\,\nu > \eta\}$, exploiting a spectral gap.

transforms (3.3) into the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}v = (A(\lambda) - \eta)v,$$

and the two spectral sets associated with the spectral gap at $\text{Re}\,\nu = \eta$ for the matrix $A(\lambda)$ become the stable and unstable spectral sets for the matrix $A(\lambda) - \eta$. Thus, the results stated above are applicable to any two spectral sets of $A(\lambda)$ of the form $\{\nu \in \text{spec}(A(\lambda));\, \text{Re}\,\nu < \eta\}$ and $\{\nu \in \text{spec}(A(\lambda));\, \text{Re}\,\nu > \eta\}$ (assuming, of course, that $\text{Re}\,\nu \neq \eta$ for every $\nu \in \text{spec}(A(\lambda))$). Note that the transformation (3.9) changes only the length of vectors but not their direction. In particular, subspaces of solutions are not changed.

In summary, we may wish to replace the condition $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ that appears in Definition 3.1 by the weaker condition $\kappa^{\mathrm{s}} < \kappa^{\mathrm{u}}$. Using the transformation (3.9) for an appropriate $\eta$, we see that all the results mentioned above are also true under this weaker condition, i.e. for arbitrary spectral gaps.

## 3.3 Spectrum and Fredholm properties

We consider the family of operators

$$\mathcal{T}(\lambda)\ :\qquad \mathcal{D} \longrightarrow \mathcal{H}, \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u$$

with parameter $\lambda$. We are interested in characterizing those $\lambda$ for which the operator $\mathcal{T}(\lambda) : \mathcal{D} \to \mathcal{H}$ is not invertible. The set of all such $\lambda$ is the spectrum of the linearization $\mathcal{L}$ about the travelling wave. We emphasize that the spectrum of the individual operators $\mathcal{T}(\lambda) : \mathcal{D} \to \mathcal{H}$, for fixed $\lambda$, is of no interest to us.

**Definition 3.2 (Spectrum)** *We say that $\lambda$ is in the spectrum $\Sigma$ of $\mathcal{T}$ if $\mathcal{T}(\lambda)$ is not invertible, i.e. if the inverse operator does not exist or is not bounded. We say that $\lambda \in \Sigma$ is in the point spectrum $\Sigma_{\mathrm{pt}}$ of $\mathcal{T}$ or, alternatively, that $\lambda \in \Sigma$ is an eigenvalue of $\mathcal{T}$ if $\mathcal{T}(\lambda)$ is a Fredholm operator with index zero. The complement $\Sigma \setminus \Sigma_{\mathrm{pt}} =: \Sigma_{\mathrm{ess}}$ is called the essential spectrum. The complement of $\Sigma$ in $\mathbb{C}$ is the resolvent set of $\mathcal{T}$.*

Recall that an operator $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is said to be a Fredholm operator if $\mathrm{R}(\mathcal{L})$ is closed in $\mathcal{Y}$, and the dimension of $\mathrm{N}(\mathcal{L})$ and the codimension of $\mathrm{R}(\mathcal{L})$ are both finite. The difference $\dim \mathrm{N}(\mathcal{L}) - \operatorname{codim} \mathrm{R}(\mathcal{L})$ is called the Fredholm index of $\mathcal{L}$. It is a measure for the solvability of $\mathcal{L}x = y$ for a given $y \in \mathcal{Y}$. Fredholm operators are amenable to a standard perturbation theory using Liapunov–Schmidt reduction. If $\mathcal{L}_\epsilon : \mathcal{X} \to \mathcal{Y}$ denotes a Fredholm operator that depends continuously on $\epsilon \in \mathbb{R}$ in the operator norm, then Liapunov–Schmidt reduction replaces the equation

$$\mathcal{L}_\epsilon u = 0$$

by a reduced equation of the form

$$\tilde{\mathcal{L}}_\epsilon u = 0, \qquad \tilde{\mathcal{L}}_\epsilon : \mathrm{N}(\mathcal{L}_0) \longrightarrow \mathrm{R}(\mathcal{L}_0)^\perp$$

that is valid for $\epsilon$ close to zero, where $\mathrm{R}(\mathcal{L}_0)^\perp$ is a complement of $\mathrm{R}(\mathcal{L}_0)$. Note that both spaces appearing in the above equation are finite-dimensional. We refer to [180, Chapter 3] and [75, Chapters I.3 and VII] for introductions to Liapunov–Schmidt reduction.

For any $\lambda$ in the point spectrum of $\mathcal{T}$, we define the multiplicity of $\lambda$ as follows. Recall that $A(\xi;\lambda)$ is of the form $A(\xi;\lambda) = \tilde{A}(\xi) + \lambda B(\xi)$. Suppose that $\lambda$ is in the point spectrum of $\mathcal{T}$, with

$$\mathcal{T}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - \tilde{A}(\xi) - \lambda B(\xi),$$

such that $\mathrm{N}(\mathcal{T}(\lambda)) = \mathrm{span}\{u_1(\cdot)\}$. We say that $\lambda$ has multiplicity $\ell$ if functions $u_j \in \mathcal{D}$ can be found for $j = 2, \ldots, \ell$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u_j(\xi) = (\tilde{A}(\xi) + \lambda B(\xi)) u_j(\xi) + B(\xi) u_{j-1}(\xi), \qquad \xi \in \mathbb{R}$$

for $j = 2, \ldots, \ell$, but so that there is no solution $u \in \mathcal{D}$ to

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = (\tilde{A}(\xi) + \lambda B(\xi)) u + B(\xi) u_\ell(\xi), \qquad \xi \in \mathbb{R}.$$

Lastly, we say that an arbitrary eigenvalue $\lambda$ of $\mathcal{T}$ has multiplicity $\ell$ if the sum of the multiplicities of a maximal set of linearly independent elements in $\mathrm{N}(\mathcal{T}(\lambda))$ is equal to $\ell$.

**Example 1 (continued)** Recall the operator $\mathcal{L} = D\partial_{\xi\xi} + c\partial_\xi + \partial_U F(Q)$ and the associated family $\mathcal{T}(\lambda)$

$$\mathcal{T}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - \tilde{A}(\xi) - \lambda B$$

with

$$\tilde{A}(\xi) = \begin{pmatrix} 0 & \mathrm{id} \\ -D^{-1}\partial_U F(Q(\xi)) & -cD^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ D^{-1} & 0 \end{pmatrix}.$$

Suppose that $\lambda$ is in the spectrum of $\mathcal{L}$ and $\mathcal{T}$. The Jordan-block structures of the operators $\mathcal{L} - \lambda$ and $\mathcal{T}(\lambda)$ are then the same, i.e. geometric and algebraic multiplicities and the length of each maximal Jordan chain are the same whether computed for $\mathcal{L} - \lambda$ or for $\mathcal{T}(\lambda)$. This justifies our definition of multiplicity for eigenvalues of $\mathcal{T}$.

It is also true that the Fredholm properties, and the Fredholm indices, of $\mathcal{L} - \lambda$ and $\mathcal{T}(\lambda)$ are the same (see e.g. [153, 157]). ■

**Remark 3.2** *The point spectrum is often defined as the set of all isolated eigenvalues with finite multiplicity, i.e. as the set $\tilde{\Sigma}_{\mathrm{pt}}$ of those $\lambda$ for which $\mathcal{T}(\lambda)$ is Fredholm with index zero, the null space of $\mathcal{T}(\lambda)$ is non-trivial, and $\mathcal{T}(\tilde{\lambda})$ is invertible for all $\tilde{\lambda}$ in a small neighbourhood of $\lambda$ (except, of course, for $\tilde{\lambda} = \lambda$).*

*The sets $\Sigma_{\mathrm{pt}}$ and $\tilde{\Sigma}_{\mathrm{pt}}$ differ in the following way. The set of $\lambda$ for which $\mathcal{T}(\lambda)$ is Fredholm with index zero is open. Take a connected component $\mathcal{C}$ of this set, then the following alternative holds. Either $\mathcal{T}(\lambda)$ is invertible for all but a discrete set of elements in $\mathcal{C}$, or else $\mathcal{T}(\lambda)$ has a non-trivial null space for all $\lambda \in \mathcal{C}$. This follows, for instance, from using the Evans function (see Section 4.1).*

The following theorem proved by Palmer relates Fredholm properties of the operator $\mathcal{T}(\lambda)$ to properties pertaining to the existence of dichotomies of (3.2)

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi;\lambda) u.$$

**Theorem 3.2** ([131, 132]) *Fix $\lambda \in \mathbb{C}$. The following statements are true.*

- *$\lambda$ is in the resolvent set of $\mathcal{T}$ if, and only if, (3.2) has an exponential dichotomy on $\mathbb{R}$.*
- *$\lambda$ is in the point spectrum $\Sigma_{\rm pt}$ of $\mathcal{T}$ if, and only if, (3.2) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with the same Morse index, $i_+(\lambda) = i_-(\lambda)$, and $\dim \mathrm{N}(\mathcal{T}(\lambda)) > 0$. In this case, denote by $P_\pm(\xi;\lambda)$ the projections of the exponential dichotomies of (3.2) on $\mathbb{R}^\pm$, then the spaces $\mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda))$ and $\mathrm{N}(\mathcal{T}(\lambda))$ are isomorphic via $u(0) \mapsto u(\cdot)$.*
- *$\lambda$ is in the essential spectrum $\Sigma_{\rm ess}$ if (3.2) either does not have exponential dichotomies on $\mathbb{R}^+$ or on $\mathbb{R}^-$, or else if it does, but the Morse indices on $\mathbb{R}^+$ and on $\mathbb{R}^-$ differ.*

As a consequence, eigenfunctions associated with elements in the point spectrum of $\mathcal{T}$ decay necessarily exponentially as $|\xi| \to \infty$.

**Remark 3.3** *To summarize the relation between Fredholm properties of $\mathcal{T}$ and exponential dichotomies of (3.2), we remark that $\mathcal{T}$ is Fredholm if, and only if, (3.2) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$. The Fredholm index of $\mathcal{T}$ is then equal to the difference $i_-(\lambda) - i_+(\lambda)$ of the Morse indices of the dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ (see [131, 132]). If $\mathcal{T}(\lambda)$ is not Fredholm, then typically the range $\mathrm{R}(\mathcal{T}(\lambda))$ of $\mathcal{T}(\lambda)$ is not closed in $\mathcal{H}$.*

Suppose that $\mathcal{T}(\lambda)$ is invertible, and denote by $\Phi^{\rm s}(\xi,\zeta;\lambda)$ and $\Phi^{\rm u}(\xi,\zeta;\lambda)$ the exponential dichotomy of (3.2) on $\mathbb{R}$. The inverse of $\mathcal{T}(\lambda)$ is then given by

$$u(\xi) = [\mathcal{T}(\lambda)^{-1}h](\xi) = \int_{-\infty}^{\xi} \Phi^{\rm s}(\xi,\zeta;\lambda)h(\zeta)\,\mathrm{d}\zeta + \int_{\infty}^{\xi} \Phi^{\rm u}(\xi,\zeta;\lambda)h(\zeta)\,\mathrm{d}\zeta.$$

If $\mathcal{T}(\lambda)$ is Fredholm with index $i$, then its range is given as follows. Consider the adjoint equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}v = -A(\xi;\lambda)^* v \tag{3.10}$$

and the associated adjoint operator

$$\mathcal{T}(\lambda)^* \;:\quad \mathcal{D} \longrightarrow \mathcal{H}, \qquad v \longmapsto -\frac{\mathrm{d}v}{\mathrm{d}\xi} - A(\cdot;\lambda)^* v \tag{3.11}$$

(note that $\mathcal{T}(\lambda)^*$ is the genuine Hilbert-space adjoint of $\mathcal{T}(\lambda)$ only when posed on the spaces (3.1)). The adjoint operator $\mathcal{T}(\lambda)^*$ is Fredholm with index $-i$. We have that $h \in \mathrm{R}(\mathcal{T}(\lambda))$ if, and only if,

$$\int_{-\infty}^{\infty} \langle \psi(\xi), h(\xi) \rangle \,\mathrm{d}\xi = 0 \tag{3.12}$$

for each $\psi \in \mathrm{N}(\mathcal{T}(\lambda)^*)$, i.e. for each bounded solution $\psi(\xi)$ of (3.10). In fact, the following remark is true.

**Remark 3.4** *Suppose that the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;\lambda)u \tag{3.13}$$

*has an exponential dichotomy on $I$ with projections $P(\xi;\lambda)$ and evolutions $\Phi^{\rm s}(\xi,\zeta;\lambda)$ and $\Phi^{\rm u}(\xi,\zeta;\lambda)$, then the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}v = -A(\xi;\lambda)^* v \tag{3.14}$$

*also has an exponential dichotomy on $I$ with projections $\tilde{P}(\xi;\lambda)$ and evolutions $\tilde{\Phi}^{\rm s}(\xi,\zeta;\lambda)$ and $\tilde{\Phi}^{\rm u}(\xi,\zeta;\lambda)$. The projections and evolutions of (3.13) and (3.14) are related via*

$$\tilde{P}(\xi;\lambda) = \mathrm{id} - P(\xi;\lambda)^*, \qquad \tilde{\Phi}^{\rm s}(\xi,\zeta;\lambda) = \Phi^{\rm u}(\zeta,\xi;\lambda)^*, \qquad \tilde{\Phi}^{\rm u}(\xi,\zeta;\lambda) = \Phi^{\rm s}(\zeta,\xi;\lambda)^*.$$

*This is a consequence of Definition 3.1 together with the following observation (see also [157, Lemma 5.1]): if $\Phi(\xi,\zeta)$ denotes the evolution of (3.13), then, upon differentiating the identity $\Phi(\xi,\zeta)\Phi(\zeta,\xi) = \mathrm{id}$ with respect to $\xi$, we see that $\tilde{\Phi}(\xi,\zeta) = \Phi(\zeta,\xi)^*$ is the evolution of (3.14).*

*In particular, $\psi \in \mathrm{N}(\mathcal{T}(\lambda)^*)$ if, and only if,*

$$\psi(0) \in \mathrm{N}(\tilde{P}_-(0;\lambda)) \cap \mathrm{R}(\tilde{P}_+(0;\lambda)) = \Big(\mathrm{N}(P_-(0;\lambda)) + \mathrm{R}(P_+(0;\lambda))\Big)^{\perp},$$

*where $P_\pm(\xi;\lambda)$ and $\tilde{P}_\pm(\xi;\lambda)$ are the projections for (3.13) and (3.14), respectively, on $I = \mathbb{R}^\pm$.*

**Remark 3.5** *Note that*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\langle u(\xi), v(\xi)\rangle = 0, \qquad \xi \in \mathbb{R}$$

*for any two solutions $u(\xi)$ and $v(\xi)$ of (3.13) and (3.14), respectively. In particular, if $u(\xi)$ and $v(\xi)$ are both bounded, and one of them converges to zero as $\xi \to \infty$ or $\xi \to -\infty$, then $\langle u(\xi), v(\xi)\rangle = 0$ for all $\xi$.*

## 3.4 Fronts, pulses and wave trains

In this section, we discuss the consequences of the above results for fronts, pulses and wave trains.

### 3.4.1 Homogeneous rest states

Suppose that the travelling wave $Q(\xi)$ is a homogeneous stationary solution, so that $Q(\xi) = Q_0 \in \mathbb{R}^n$ does not depend on $\xi$. The coefficients of the PDE linearization about $Q_0$ are constant and do not depend on $\xi$. Thus, assume that

$$A(\xi;\lambda) = A_0(\lambda) = \tilde{A}_0 + \lambda B_0$$

does not depend on $\xi$ and consider (3.2), now given by

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A_0(\lambda)u.$$

This equation has an exponential dichotomy on $\mathbb{R}$ if, and only if, $A_0(\lambda)$ is hyperbolic. In fact, if $A_0(\lambda)$ is hyperbolic, then

$$\Phi^{\mathrm{s}}(\xi,\zeta;\lambda) = \mathrm{e}^{A_0(\lambda)(\xi-\zeta)}P_0^{\mathrm{s}}(\lambda), \qquad \Phi^{\mathrm{u}}(\xi,\zeta;\lambda) = \mathrm{e}^{A_0(\lambda)(\xi-\zeta)}P_0^{\mathrm{u}}(\lambda),$$

where $P_0^{\mathrm{s}}(\lambda)$ and $P_0^{\mathrm{u}}(\lambda)$ are the spectral projections of $A_0(\lambda)$ associated with the stable and unstable spectral sets, respectively.

We have the following alternative:

- $\lambda$ is in the resolvent set of $\mathcal{T}$ if, and only if, $A_0(\lambda)$ is hyperbolic.
- $\lambda$ is in the essential spectrum $\Sigma_{\mathrm{ess}}$ if, and only if, $A_0(\lambda)$ has at least one purely imaginary eigenvalue, i.e. $\Sigma_{\mathrm{ess}} = \{\lambda \in \mathbb{C};\ \mathrm{spec}(A_0(\lambda)) \cap \mathrm{i}\mathbb{R} \neq \emptyset\}$.

In particular, the point spectrum is empty.

**Example 1 (continued)** Suppose that $Q_0$ is a homogeneous rest state. Hence,

$$A_0(\lambda) = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\lambda - \partial_U F(Q_0)) & -cD^{-1} \end{pmatrix},$$

and $\lambda$ is in the essential spectrum of $\mathcal{T}$ if, and only if,

$$d_0(\lambda,k) = \det[A_0(\lambda) - \mathrm{i}k] = 0$$

has a solution $k \in \mathbb{R}$. The function $d_0(\lambda, k)$ is often referred to as the (linear) dispersion relation. Typically, the essential spectrum consists of the union of curves $\lambda_*(k)$ in the complex plane, where $\lambda_*(k)$ is such that $d_0(\lambda_*(k), k) = 0$ for $k \in \mathbb{R}$. Alternatively, the essential spectrum can be calculated by substituting $U(\xi, t) = \mathrm{e}^{\lambda t + \mathrm{i}k\xi} U_0$ into the linear equation $U_t = \mathcal{L}_0 U$.

An interesting quantity is the group velocity

$$c_{\text{group}} = -\frac{\mathrm{d}}{\mathrm{d}k} \operatorname{Im} \lambda_*(k)$$

which is the velocity with which wave packets with Fourier spectrum centered near the frequency $k$ evolve with respect to the equation $U_t = \mathcal{L}_0 U$. We refer to [24, Section 2] for more details regarding the physical interpretation of the group velocity. ∎

### 3.4.2 Periodic wave trains

If we consider the linearization about a spatially-periodic travelling wave $Q(\xi)$ with spatial period $L$, i.e. about a wave train $Q(\xi)$ with $Q(\xi + L) = Q(\xi)$ for all $\xi$, then the coefficients of the PDE linearization have period $L$ in $\xi$. Thus, we assume that the matrix $A(\xi; \lambda)$ is periodic in $\xi$ with period $L > 0$,

$$A(\xi + L; \lambda) = A(\xi; \lambda), \qquad \xi \in \mathbb{R},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi; \lambda) u = (\tilde{A}(\xi) + \lambda B(\xi)) u \tag{3.15}$$

has periodic coefficients. By Floquet theory (see e.g. [83, Chapter IV.6]), the evolution $\Phi(\xi, \zeta; \lambda)$ of (3.15) is of the form

$$\Phi(\xi, 0; \lambda) = \Phi_{\text{per}}(\xi; \lambda) \mathrm{e}^{R(\lambda)\xi}$$

where $R(\lambda) \in \mathbb{C}^{n \times n}$ and $\Phi_{\text{per}}(\xi + L; \lambda) = \Phi_{\text{per}}(\xi; \lambda)$ for all $\xi \in \mathbb{R}$ with $\Phi_{\text{per}}(0; \lambda) = \text{id}$. Note that it is not clear whether we can choose $R(\lambda)$ to be analytic in $\lambda$ (though this is always possible locally in $\lambda$).

We have the following alternatives: The point spectrum $\Sigma_{\text{pt}}$ is empty, and

- $\lambda$ is in the resolvent set of $\mathcal{T}$ if, and only if, $\operatorname{spec}(R(\lambda)) \cap \mathrm{i}\mathbb{R} = \emptyset$, i.e. if $\Phi(L, 0; \lambda)$ has no purely imaginary Floquet exponent (or, equivalently, if $\Phi(L, 0; \lambda)$ has no spectrum on the unit circle).
- $\Sigma_{\text{ess}} = \{\lambda \in \mathbb{C};\ \operatorname{spec}(R(\lambda)) \cap \mathrm{i}\mathbb{R} \neq \emptyset\} = \{\lambda \in \mathbb{C};\ \operatorname{spec}(\Phi(L, 0; \lambda)) \cap S^1 \neq \emptyset\}$.

Consequently, $\lambda$ is in the essential spectrum if, and only if, the boundary-value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} u &= A(\xi; \lambda) u, \qquad 0 < \xi < L \\ u(L) &= \mathrm{e}^{\mathrm{i}\gamma} u(0) \end{aligned} \tag{3.16}$$

has a solution $u(\xi)$ for some $\gamma \in \mathbb{R}$. This is the case precisely if $\mathrm{i}\gamma$ is a purely imaginary Floquet exponent of $\Phi(L, 0; \lambda)$.

The approach via Floquet theory is also applicable in higher space dimensions [163, 164], then often referred to as decomposition into Bloch waves, and we refer to [23, 118, 119] for generalizations and applications to Turing patterns.

Suppose that the wave train is found as a periodic solution $q(\xi)$ to

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = f(u, c)$$

and that (3.15) is given by

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = (\partial_u f(q(\xi), c) + \lambda B)u$$

with

$$\partial_c f(u(\xi), c) = -Bu'(\xi).$$

As a consequence, $\lambda = 0$ is contained in the essential spectrum, since $q'(\xi)$ satisfies (3.16) for $\gamma = 0$. Furthermore, spatially-periodic wave trains with period $L$ typically exist for any period $L$, in a certain range, for a wave speed $c(L)$ that depends on $L$ (see e.g. [33, 80, 107]). It is not hard to verify, using the equations above, that

$$\frac{\mathrm{d}}{\mathrm{d}L}c(L) = -c_{\text{group}} = \frac{\mathrm{d}}{\mathrm{d}\gamma}\,\mathrm{Im}\,\lambda(\gamma)\Big|_{\gamma=0}$$

where $\lambda(\gamma)$ denotes the solution to (3.16) that satisfies $\lambda(0) = 0$. The group velocity at $\lambda = 0$ is therefore related to the nonlinear dispersion relation $c = c(L)$ that relates wave speed and wavelength of the wave trains. We refer to [24, 117] for the physical interpretation of the group velocity.

We remark that, for each fixed $\gamma \in \mathbb{R}$, the multiplicity of an eigenvalue $\lambda$ to (3.16) can again be defined as in Section 3.3 by using Jordan chains

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u_j = A(\xi;\lambda)u_j + B(\xi)u_{j-1}, \qquad u_j(L) = \mathrm{e}^{\mathrm{i}\gamma}u_j(0)$$

(see [68]). These eigenvalues, counted with their multiplicity, can be sought as zeros of the Evans function

$$D_{\text{per}}(\gamma,\lambda) = \det[\mathrm{e}^{\mathrm{i}\gamma} - \Phi(L,0;\lambda)]. \tag{3.17}$$

It has been proved in [68] that, for fixed $\gamma \in \mathbb{R}$, $\lambda_*$ is a solution to (3.16) with multiplicity $\ell$ if, and only if, $\lambda_*$ is a zero of $D_{\text{per}}(\gamma,\lambda)$ of order $\ell$.

### 3.4.3 Fronts

Suppose that the travelling wave $Q(\xi)$ is a front, so that the limits

$$\lim_{\xi\to\pm\infty} Q(\xi) = Q_\pm \in \mathbb{R}^N$$

exist. The vectors $Q_\pm$ are homogeneous stationary solutions to the underlying PDE, and we refer to $Q_\pm$ as the asymptotic rest states. Thus, the coefficients of the underlying PDE linearization have limits as $\xi \to \pm\infty$. We assume that there are $n \times n$ matrices $\tilde{A}_\pm$ and $B_\pm$ such that

$$\lim_{\xi\to\pm\infty} \tilde{A}(\xi) = \tilde{A}_\pm, \qquad \lim_{\xi\to\pm\infty} B(\xi) = B_\pm$$

and define

$$A_\pm(\lambda) = \tilde{A}_\pm + \lambda B_\pm.$$

The existence of exponential dichotomies for the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;\lambda)u = (\tilde{A}(\xi) + \lambda B(\xi))u \tag{3.18}$$

on $\mathbb{R}^\pm$ is related to the hyperbolicity of the asymptotic matrices $A_\pm(\lambda)$. The next theorem rephrases the statement of Theorem 3.1.

**Theorem 3.3 ([36, Chapter 6])** *Fix $\lambda \in \mathbb{C}$. Equation (3.18) has an exponential dichotomy on $\mathbb{R}^+$ if, and only if, the matrix $A_+(\lambda)$ is hyperbolic. In this case, the Morse index $i_+(\lambda)$ is equal to the dimension $\dim E^{\mathrm{u}}_+(\lambda)$ of the generalized unstable eigenspace $E^{\mathrm{u}}_+(\lambda)$ of $A_+(\lambda)$. This statement is also true on $\mathbb{R}^-$ with $A_+(\lambda)$ replaced by $A_-(\lambda)$.*

*Lastly, (3.18) has an exponential dichotomy on $\mathbb{R}$ if, and only if, it has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with projections $P_\pm(\xi;\lambda)$ such that $\mathrm{N}(P_-(0;\lambda)) \oplus \mathrm{R}(P_+(0;\lambda)) = \mathbb{C}^n$; this requires in particular that the Morse indices $i_+(\lambda)$ and $i_-(\lambda)$ are equal.*

With a slight abuse of notation, we will refer to the number of unstable eigenvalues of a hyperbolic $n \times n$ matrix $A$, counted with multiplicity, as its Morse index. We observe that, using this notation, the Morse indices of the asymptotic matrices $A_\pm(\lambda)$ are equal to the Morse indices $i_\pm(\lambda)$ of the exponential dichotomies on $\mathbb{R}^\pm$ by the above theorem.

Note that $\mathcal{T}(\lambda)$ is Fredholm with index zero if, and only if, the number of linearly independent solutions to (3.18) that decay as $\xi \to -\infty$ and the number of solutions that decay as $\xi \to \infty$ add up to the dimension $n$ of $\mathbb{C}^n$.

As a consequence of Theorems 3.2 and 3.3, we have the following options:

- $\lambda$ is in the resolvent set of $\mathcal{T}$ if, and only if, $A_\pm(\lambda)$ are both hyperbolic with the same Morse index $i_+(\lambda) = i_-(\lambda)$ such that the projections $P_\pm(\xi;\lambda)$ of the exponential dichotomies of (3.18) on $I = \mathbb{R}^\pm$ satisfy $\mathrm{N}(P_-(0;\lambda)) \oplus \mathrm{R}(P_+(0;\lambda)) = \mathbb{C}^n$.
- $\lambda$ is in the point spectrum $\Sigma_{\mathrm{pt}}$ if, and only if, the asymptotic matrices $A_\pm(\lambda)$ are both hyperbolic with identical Morse index $i_+(\lambda) = i_-(\lambda)$ such that the projections $P_\pm(\xi;\lambda)$ of the exponential dichotomies of (3.18) on $I = \mathbb{R}^\pm$ satisfy $\mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda)) \neq \{0\}$.
- $\lambda$ is in the essential spectrum $\Sigma_{\mathrm{ess}}$ if either at least one of the two asymptotic matrices $A_\pm(\lambda)$ is not hyperbolic (so that $\lambda$ is in the essential spectrum of one or both rest states $Q_\pm$) or else if $A_+(\lambda)$ and $A_-(\lambda)$ are both hyperbolic but their Morse indices differ, so that $i_+(\lambda) \neq i_-(\lambda)$.

The reason that the boundary of the essential spectrum depends only on the asymptotic rest states $Q_\pm$ is related to the fact that the operators $\mathcal{T}(\lambda)$ and

$$\hat{\mathcal{T}}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - \hat{A}(\cdot;\lambda)$$

with

$$\hat{A}(\xi;\lambda) = \begin{cases} A_-(\lambda) & \text{for } \xi < 0 \\ A_+(\lambda) & \text{for } \xi \geq 0 \end{cases}$$

differ only by a relatively compact operator (see [85, Appendix to Section 5: Theorem A.1 and Exercise 2]).

Typically, the essential spectrum of fronts contains open sets in the complex plane, namely regions where $\mathcal{T}(\lambda)$ is Fredholm with non-zero index $i_-(\lambda) - i_+(\lambda) \neq 0$. Note that $\lambda = 0$ is always contained in the spectrum with eigenfunction $Q'(\xi)$.

**Example 1 (continued)** Suppose that $Q(\xi)$ is a front connecting the asymptotic rest states $Q_\pm$, so that

$$A_\pm(\lambda) = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\lambda - \partial_U F(Q_\pm)) & -cD^{-1} \end{pmatrix}.$$

Thus, $\lambda$ is in the essential spectrum of $\mathcal{T}$ if either $\lambda$ is in the essential spectrum of $Q_+$ or $Q_-$ (see Section 3.4.1) or else if the Morse indices $i_-(\lambda)$ and $i_+(\lambda)$, i.e. the number of unstable eigenvalues of $A_\pm(\lambda)$, differ. ∎

### 3.4.4 Pulses

Suppose that the travelling wave $Q(\xi)$ is a pulse such that

$$\lim_{|\xi|\to\infty} Q(\xi) = Q_0 \in \mathbb{R}^N.$$

In other words, a pulse is a front that connects to the same rest state $Q_0$ as $\xi \to \pm\infty$. Thus, we assume that there are $n \times n$ matrices $\tilde{A}_0$ and $B_0$ such that

$$\lim_{|\xi|\to\infty} \tilde{A}(\xi) = \tilde{A}_0, \qquad \lim_{|\xi|\to\infty} B(\xi) = B_0$$

and define

$$A_0(\lambda) = \tilde{A}_0 + \lambda B_0.$$

This is a special case of the situation for fronts considered above. The main difference is that the Morse indices at $\xi = +\infty$ and $\xi = -\infty$ are always the same. As a consequence, the operator $\mathcal{T}(\lambda)$ is either not Fredholm or is Fredholm with index zero. We have the following statement.

- $\lambda$ is in the resolvent set of $\mathcal{T}$ if, and only if, the asymptotic matrix $A_0(\lambda)$ is hyperbolic, and the projections $P_\pm(\xi;\lambda)$ of the exponential dichotomies of (3.18) on $\mathbb{R}^\pm$ satisfy $\mathrm{N}(P_-(0;\lambda)) \oplus \mathrm{R}(P_+(0;\lambda)) = \mathbb{C}^n$.
- $\lambda$ is in the point spectrum $\Sigma_{\mathrm{pt}}$ if, and only if, $A_0(\lambda)$ is hyperbolic, and the projections $P_\pm(\xi;\lambda)$ of the exponential dichotomies of (3.18) on $\mathbb{R}^\pm$ satisfy $\mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda)) \neq \{0\}$.
- $\lambda$ is in the essential spectrum $\Sigma_{\mathrm{ess}}$ if the asymptotic matrix $A_0(\lambda)$ is not hyperbolic, i.e. if $\lambda$ is in the essential spectrum of the asymptotic rest state $Q_0$.

Again, $\lambda = 0$ is always contained in the spectrum with eigenfunction $Q'(\xi)$ (see Example 1).

### 3.4.5 Fronts connecting periodic waves

Similar results are true for fronts that connect spatially-periodic waves to each other or to homogeneous rest states. The essential spectrum of such fronts is determined by the essential spectra of the asymptotic wave trains or rest states and their Morse indices: The essential spectra of the asymptotic wave trains or rest states has been computed in Sections 3.4.2 and 3.4.1 above. Exponential dichotomies for the asymptotic linearizations generate exponential dichotomies for the linearization about the front, and vice versa, by Theorem 3.1. We omit the details.

## 3.5 Absolute and convective instability

In this section, we report on absolute and convective instabilities that are related to the essential spectrum of a travelling wave. We refer to [16, 24, 135, 162, 155] for further details and more background, and also to [140] for instructive examples. Related phenomena for matrices (via discretizations of PDE operators) are reviewed in [175].

Suppose that we consider a travelling wave that has essential spectrum in the right half-plane, so that the wave is unstable. Such an instability can manifest itself in different ways. The physics literature distinguishes between two different kinds of instability, namely absolute and convective instabilities. An absolute instability occurs if perturbations grow in time at every fixed point in the domain (see Figure 4a). Convective instabilities are characterized by the fact that, even though the overall norm of the perturbation grows in time, perturbations decay locally at every fixed point in the unbounded domain; in other words, the growing perturbation is transported, or convected, towards infinity (see Figure 4b).[2]

[2] Note that the difference between absolute and convective instabilities depends crucially on the choice of the spatial coordinate system: changing to a moving frame can turn a convective instability into an absolute instability

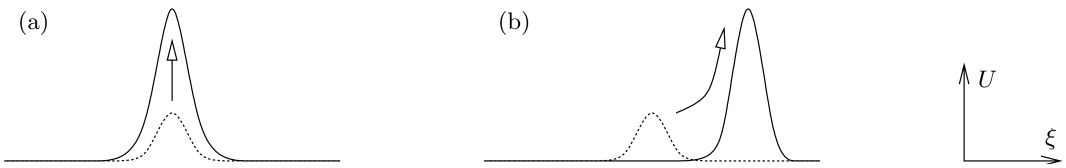


**Figure 4** The dotted waves are the initial data $U_0(\xi)$ to the linearized equation $U_t = \mathcal{L}U$, whereas the solid waves represent the solution $U(\xi,t)$ at a fixed positive time $t$. In (a), an absolute instability is shown: the solution grows without bounds at each given point $\xi$ in space as $t \to \infty$. In (b), a convective instability is shown: the solution $U(\xi,t)$ grows but also travels towards $\xi = +\infty$; $U(\xi,t)$ actually decays for each fixed value of $\xi$ as $t \to \infty$. The operator $\mathcal{L}$ would then have stable spectrum in the norm $\|\cdot\|_\eta$ for a certain $\eta > 0$.

We outline how absolute and convective instabilities can be captured mathematically on the unbounded domain $\mathbb{R}$. Suppose that the linearization of the PDE about a pulse, say, is given by the operator $\mathcal{L}$, acting on the space $L^2(\mathbb{R})$ with norm $\|\cdot\|$. To describe convective instabilities, it is convenient to introduce exponential weights [162]: for a given real number $\eta$, define the norm $\|\cdot\|_\eta$ by

$$\|U\|_\eta^2 = \int_{-\infty}^{\infty} |\mathrm{e}^{-\eta\xi} U(\xi)|^2 \,\mathrm{d}\xi, \tag{3.19}$$

and denote by $L^2_\eta(\mathbb{R})$, equipped with the norm $\|\cdot\|_\eta$, the space of functions $U(\xi)$ for which $\|U\|_\eta < \infty$. Note that the norms $\|\cdot\|_\eta$ for different values of $\eta$ are not equivalent to each other. We consider $\mathcal{L}$ as an operator on $L^2_\eta(\mathbb{R})$ and compute its spectrum using the new norm $\|\cdot\|_\eta$ for appropriate values of $\eta$. The key is that, for $\eta > 0$, the norm $\|\cdot\|_\eta$ penalizes perturbations at $-\infty$, while it tolerates perturbations (which may in fact grow exponentially with any rate less than $\eta$) at $+\infty$. Thus, if an instability is of transient nature, so that it manifests itself by modes that travel towards $+\infty$, then the essential spectrum calculated in the norm $\|\cdot\|_\eta$ should move to the left as $\eta > 0$ increases. Indeed, as the perturbations travel towards $+\infty$, they are multiplied by the weight $\mathrm{e}^{-\eta\xi}$ which is small as $\xi \to \infty$ and therefore reduces their growth or even causes them to decay. Exponential weights have been used to study a variety of problems posed on the real line such as reaction-diffusion operators [162], conservative systems such as the KdV equation [135, 136], and generalized Kuramoto–Sivashinsky equations that describe thin films [30, 31]. They have also been applied to spiral waves [159] on the plane. Absolute instabilities occur if the spectrum cannot be stabilized by any choice of $\eta$. Conditions for the presence of an absolute instability were derived for homogeneous rest states in [24, Section 2] and for wave trains in [16]. We also refer to [155] for related phenomena.

Introducing the weight (3.19) has the effect that the operator $\mathcal{T}$ given by

$$\mathcal{T}(\lambda): \qquad \mathcal{D} \longrightarrow \mathcal{H}, \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u$$

is replaced by the operator

$$\mathcal{T}^\eta(\lambda): \qquad \mathcal{D} \longrightarrow \mathcal{H}, \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - [A(\cdot;\lambda) - \eta]u$$

upon using the transformation (3.9). In particular, the essential spectrum of the operator $\mathcal{L}$ in the weighted space can be computed by applying the theory outlined in the previous sections to the operator $\mathcal{T}^\eta(\lambda)$, rather than to the operator $\mathcal{T}(\lambda)$. These arguments also apply to fronts instead of pulses: it is then, however, often necessary to consider different exponents for $\xi < 0$ and for $\xi > 0$ in the weight function to accommodate the different asymptotic matrices $A_\pm(\lambda)$. We refer to [155] for more details and references.

# 4 The Evans function

We have seen that the spectrum of $\mathcal{T}$ is the union of the essential spectrum $\Sigma_{\rm ess}$ and the point spectrum $\Sigma_{\rm pt}$. For pulses and fronts, the essential spectrum can be calculated by solving the linear dispersion relation of the asymptotic rest states (see Section 3.4). In this section, we review the Evans function which provides a tool for locating the point spectrum [2, 134]. The Evans function can also be used to locate the essential spectrum of wave trains [68] (see Section 3.4.2).

## 4.1 Definition and properties

Consider the eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;\lambda)u, \qquad u \in \mathbb{C}^n, \qquad \xi \in \mathbb{R}. \tag{4.1}$$

Since we are interested in locating the point spectrum, we assume that $\lambda$ is not in the essential spectrum $\Sigma_{\rm ess}$ of $\mathcal{T}$ (see, however, Section 4.3). Owing to Theorem 3.2, equation (4.1) therefore has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ with projections $P_+(\xi;\lambda)$ and $P_-(\xi;\lambda)$, respectively, and the Morse indices $\dim \mathrm{N}(P_+(0;\lambda)) = \dim \mathrm{N}(P_-(0;\lambda))$ are the same.

Recall from Definition 3.1 that $\mathrm{R}(P_+(0;\lambda))$ contains all initial conditions $u(0)$ whose associated solutions $u(\xi)$ of (4.1) decay exponentially as $\xi \to \infty$. Analogously, $\mathrm{N}(P_-(0;\lambda))$ consists of all initial conditions $u(0)$ whose associated solutions $u(\xi)$ decay exponentially as $\xi \to -\infty$. In particular, owing to Theorem 3.2, we have that $\lambda \in \Sigma_{\rm pt}$ if, and only if,

$$\mathrm{N}(\mathcal{T}(\lambda)) \cong \mathrm{N}(P_-(0;\lambda)) \cap \mathrm{R}(P_+(0;\lambda)) \neq \{0\}.$$

Any eigenfunction $u(\xi)$ is a bounded solution to the eigenvalue problem (4.1): $u(0)$ should therefore lie in $\mathrm{R}(P_+(0;\lambda))$, so that $u(\xi)$ is bounded for $\xi > 0$, and in $\mathrm{N}(P_-(0;\lambda))$, so that $u(\xi)$ is bounded for $\xi < 0$. The Evans function $D(\lambda)$ is designed to locate non-trivial intersections of $\mathrm{R}(P_+(0;\lambda))$ and $\mathrm{N}(P_-(0;\lambda))$.

Let $\Omega$ be a simply-connected subset of $\mathbb{C} \setminus \Sigma_{\rm ess}$. Note that, in most applications, the essential spectrum $\Sigma_{\rm ess}$ will be contained in the left half-plane; otherwise the wave is already unstable. The set $\Omega$ of interest is then the connected component $\Omega_\infty$ of $\mathbb{C} \setminus \Sigma_{\rm ess}$ that contains the right half-plane.

The Morse index $\dim \mathrm{N}(P_-(0;\lambda)) = \dim \mathrm{N}(P_+(0;\lambda))$ is constant for $\lambda \in \Omega$, see Remark 3.1, and we denote it by $k$. We choose ordered bases $[u_1(\lambda), \dots, u_k(\lambda)]$ and $[u_{k+1}(\lambda), \dots, u_n(\lambda)]$ of $\mathrm{N}(P_-(0;\lambda))$ and $\mathrm{R}(P_+(0;\lambda))$, respectively. On account of [100, Chapter II.4.2], we can choose these basis vectors in an analytic fashion, so that $u_j(\lambda)$ depends analytically on $\lambda \in \Omega$ for $j = 1, \dots, n$. We can also choose these basis vectors to be real whenever $\lambda$ is real (recall that we assumed that the matrices $\tilde{A}(\xi)$ and $B(\xi)$ are real).

**Definition 4.1 (The Evans function)** *The Evans function is defined by*

$$D(\lambda) = \det[u_1(\lambda), \dots, u_n(\lambda)].$$

An immediate consequence of this definition is that $D(\lambda) = 0$ if, and only if, $\lambda$ is an eigenvalue of $\mathcal{T}$.

Note that the Evans function depends on the choice of the basis vectors $u_j(\lambda)$. Any two Evans functions, however, differ only by a product with an analytic function that never vanishes; this factor is given by the determinants of the transformation matrices that describe the change of bases. Since this ambiguity in the construction is of no consequence, we sometimes use, with an abuse of notation, the shortcut

$$D(\lambda) = \mathrm{N}(P_-(0;\lambda)) \wedge \mathrm{R}(P_+(0;\lambda))$$

to denote the Evans function associated with the subspaces $\mathrm{N}(P_-(0;\lambda))$ and $\mathrm{R}(P_+(0;\lambda))$, even though the above construction is not unique. Note that we could make the Evans function unique by, for instance, fixing an orientation for both subspaces $\mathrm{N}(P_-(0;\lambda))$ and $\mathrm{R}(P_+(0;\lambda))$ and by choosing oriented, orthonormal bases.

**Theorem 4.1 ([54, 2, 70, 134])** *The Evans function $D(\lambda)$ is analytic in $\lambda \in \Omega$ and has the following properties.*

- *$D(\lambda) \in \mathbb{R}$ whenever $\lambda \in \mathbb{R} \cap \Omega$.*
- *$D(\lambda) = 0$ if, and only if, $\lambda$ is an eigenvalue of $\mathcal{T}$.*
- *The order of $\lambda_*$ as a zero of the Evans function $D(\lambda)$ is equal to the algebraic multiplicity of $\lambda_*$ as an eigenvalue of $\mathcal{T}$ (see Section 3.3).*

Here, the order of $\lambda_*$ as a zero of $D(\lambda)$ is the unique integer $m \geq 0$ for which

$$\frac{\mathrm{d}^j D}{\mathrm{d}\lambda^j}(\lambda_*) = 0 \quad (\text{for } j = 0, \ldots, m-1), \qquad \frac{\mathrm{d}^m D}{\mathrm{d}\lambda^m}(\lambda_*) \neq 0.$$

The reason why the Evans function counts the algebraic multiplicity of eigenvalues is related to the fact that, if we denote by $\Phi(\xi, \zeta; \lambda)$ the evolution of

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = (\tilde{A}(\xi) + \lambda B(\xi))u,$$

then, for each $u_0 \in \mathbb{C}^n$, the derivative $\partial_\lambda \Phi(\xi, 0; \lambda) u_0$ is a particular solution to

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = (\tilde{A}(\xi) + \lambda B(\xi))u + B(\xi)\Phi(\xi, 0; \lambda)u_0.$$

This equation is precisely the equation that determines the algebraic multiplicity of $\lambda$ (see Section 3.3).

Thus, the Evans function locates eigenvalues of $\mathcal{T}$ with their algebraic multiplicity. As we have seen in Example 1 and in Section 3.4, the Evans function typically vanishes at $\lambda = 0$, so that $D(0) = 0$: the derivative $Q_\xi(\xi)$ of the travelling wave $Q(\xi)$ generates the eigenvalue $\lambda = 0$.

Note that, because of analyticity, the Evans function $D(\lambda)$ either vanishes identically in $\Omega$ or else it has a discrete set of zeros with finite order corresponding to isolated eigenvalues of $\mathcal{T}$ with finite multiplicity. This proves the statement in Remark 3.2.

## 4.2 The computation of the Evans function, and applications

In general, it is difficult to calculate the Evans function explicitly for a given PDE.

One class of PDEs for which the Evans function can often be computed is integrable PDEs for which Inverse Scattering Theory is available. Examples for which Evans functions have been calculated include the Korteweg–de Vries and the modified Korteweg–de Vries equation [135], Boussinesq-type equations [6, 136], the nonlinear Schrödinger equation [97], and the fourth-order PDE [3] from Example 2.

In singularly perturbed reaction-diffusion equations,

$$\begin{aligned} \epsilon\tau u_t &= \epsilon^2 u_{xx} + f(u,v) \\ v_t &= \delta^2 v_{xx} + g(u,v), \end{aligned} \tag{4.2}$$

with $\epsilon > 0$ small, travelling waves are often constructed by piecing or gluing several singular waves together using, for instance, geometric singular perturbation theory (see [91] for a review) or matched asymptotic expansions (see e.g. [176, 112]). These singular waves are travelling waves of (4.2) in the limit $\epsilon \to 0$

in various different scalings of the spatial variable $x$ or $\xi$. In particular, the singular waves are stationary solutions of certain scalar reaction-diffusion equations, and their stability properties follow immediately from Sturm–Liouville theory (see e.g. [83, Chapter XI]). The issue is to determine the stability properties of the travelling wave to the full equation (4.2) for $\epsilon > 0$. This can often be achieved using the Evans function: We refer to [90, 185] for results proving the stability of the fast pulses to the FitzHugh–Nagumo equation and to [15, 47, 70, 88, 127, 128, 129, 142] and the references therein for various other results related to the stability of waves to singularly perturbed equations of the form (4.2). One particularly useful argument is provided by the so-called "elephant-trunk lemma" (see [71]) that shows that the Evans function of the full equation (4.2) is often close to the product of the Evans functions for each of the two equations appearing in (4.2), properly scaled and appropriately evaluated (see [2, 71] for details and applications).

The Evans function can be used to test for instability using a parity-type argument. Suppose that the essential spectrum $\Sigma_{\mathrm{ess}}$ is contained in the open left half-plane; otherwise, the wave is unstable. The idea is that the Evans function $D(\lambda)$ is defined and real for all real $\lambda \geq 0$. Suppose that the Evans function is positive, $D(\lambda) \geq \delta > 0$, for all sufficiently large real $\lambda$. Note also that $D(0) = 0$ for any non-trivial travelling wave because of translation invariance. Hence, if $D'(0) < 0$, then at least one $\lambda_* > 0$ exists with $D(\lambda_*) = 0$, and the wave is unstable, since $\lambda_*$ is in the point spectrum $\Sigma_{\mathrm{pt}}$. The key is that there are computable expressions for the derivatives of the Evans function with respect to $\lambda$ and also with respect to parameters that appear in the underlying PDE (see [134, 146, 95] and Section 4.2.1). Also, the limiting behaviour of the Evans function $D(\lambda)$ as $\lambda \to \infty$ can often be determined (see Section 4.2.2). The above parity argument has been used to derive instability criteria for solitary waves in Hamiltonian PDEs [134, 21, 22], for multi-bump pulses to reaction-diffusion equations [4, 5, 122, 187], and for viscous shocks in conservation laws [72].

Lastly, the Evans function is also useful when computing the stability of solitary waves under perturbations. Suppose, for instance, that $\lambda = 0$ is a zero of $D(\lambda)$ with order $m > 1$. This typically occurs when the underlying PDE has continuous symmetries such as a phase invariance or Galilean invariance. Examples are the nonlinear Schrödinger equation with $m = 4$ [183] and the complex Ginzburg–Landau equation with $m = 2$ [177]. If some of these symmetries are broken upon adding perturbations, then some of the eigenvalues at $\lambda = 0$ may move away, and it is necessary to locate these additional discrete eigenvalues to determine stability. Using expressions for the derivatives of the Evans function with respect to $\lambda$ and the perturbation parameters as provided by [134, 146, 95], the Evans function can be expanded in a Taylor series, and its zeros near $\lambda = 0$ can, in principle, be computed. We refer to [94] where this approach has been carried out for the perturbation of the nonlinear Schrödinger equation to the complex Ginzburg–Landau equation. We emphasize that the perturbed eigenvalues can often be computed more efficiently using Liapunov–Schmidt reduction. The reason is that, near a given isolated eigenvalue $\lambda_*$ in the point spectrum of $\mathcal{T}$, the Evans function $D(\lambda)$ is essentially equal to the determinant of the underlying PDE operator $\mathcal{L}$ restricted to the generalized eigenspace of $\lambda_*$. Thus, upon adding a perturbation to the operator $\mathcal{L}$, the polynomial $D(\lambda)$ is perturbed, and we need to compute its zeros: in general, this is a difficult task. It is often far easier to investigate directly the matrix that represents $\mathcal{L}$ restricted to the eigenspace of $\lambda_*$. In other words, it is often easier to compute the eigenvalues of a matrix directly, using properties of the matrix, rather than finding zeros of the characteristic polynomial which may hide these properties. For instance, symmetries can often be exploited more efficiently. We refer to [106, 151] where Liapunov–Schmidt reduction rather than the Evans function has been utilized.

Below, we give an expression for the derivative $D'(0)$ of the Evans function $D(\lambda)$ at $\lambda = 0$ and explore the asymptotic behaviour of $D(\lambda)$ for $\lambda \to \infty$ in more detail using Example 1.

### 4.2.1 The derivative $\mathbf{D}'(\mathbf{0})$

We assume that $\lambda = 0$ is contained in the point spectrum of $\mathcal{T}$, with geometric multiplicity one. In particular, we assume that $\mathcal{T}(0)$ is Fredholm with index zero. We denote the non-zero eigenfunction of $\lambda = 0$ by $\varphi(\xi)$.[3] Hence, $\varphi(\xi)$ is the unique bounded solution (unique up to constant multiples) of

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;0)u,$$

and we have

$$\operatorname{span}\{\varphi(0)\} = \mathrm{N}(P_-(0;0)) \cap \mathrm{R}(P_+(0;0)).$$

We choose the ordered bases $[u_1(\lambda), \dots, u_k(\lambda)]$ and $[u_{k+1}(\lambda), \dots, u_n(\lambda)]$ of $\mathrm{N}(P_-(0;\lambda))$ and $\mathrm{R}(P_+(0;\lambda))$, respectively, in the definition of the Evans function in such a fashion that

$$u_1(0) = \varphi(0), \qquad u_{k+1}(0) = \varphi(0).$$

Owing to Remark 3.4 and to the fact that $\mathcal{T}(0)$ is Fredholm with index zero, the adjoint equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = -A(\xi;0)^* u$$

also has a unique bounded solution $\psi(\xi)$. The derivative of the Evans function $D(\lambda)$ at $\lambda = 0$ is then given by

$$D'(0) = \frac{1}{|\psi(0)|^2} \det[\psi(0), u_2(0), \dots, u_k(0), \varphi(0), u_{k+2}(0), \dots, u_n(0)] \times \int_{-\infty}^{\infty} \langle \psi(\xi), B(\xi)\varphi(\xi)\rangle \,\mathrm{d}\xi. \tag{4.3}$$

This expression can be evaluated once $\psi(\xi)$ and $\varphi(\xi)$ are known. Note that the right-hand side of (4.3) is a product of two terms: The non-zero determinant measures only the orientation of the basis in the brackets; note that $\psi(0)$ is perpendicular to the vectors $u_j(0)$ at $\lambda = 0$ as a consequence of Remark 3.5. The integral decides whether $\lambda = 0$ has higher algebraic multiplicity. Indeed, the integral is equal to zero if, and only if, $B(\cdot)\varphi(\cdot)$ is contained in the range of $\mathcal{T}(0)$, see (3.12), which is the case precisely when $\lambda = 0$ has algebraic multiplicity larger than one. This observation illustrates one relation between the statements in Section 3.3 and the Evans function.

If the underlying PDE is Hamiltonian, then the derivative $D'(0)$ is typically zero, and an expression for the second-order derivative $D''(0)$ is needed. In [134], the quantity $D''(0)$ has been related to the derivative of the momentum functional[4] with respect to the wave speed $c$. We refer to [134] for the details and to [21, 22] for extensions that utilize a multi-symplectic formulation of the underlying Hamiltonian PDE.

### 4.2.2 The asymptotic behaviour of $\mathbf{D}(\lambda)$ as $\lambda \to \infty$

We illustrate the typical behaviour of the Evans function $D(\lambda)$ in the limit as $\lambda \to \infty$ using Example 1.

**Example 1 (continued)** Consider the eigenvalue problem[5]

$$\begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} V \\ D^{-1}(\lambda U - \partial_U F(Q(\xi))U - cV) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\lambda - \partial_U F(Q(\xi))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{4.4}$$

[3] If $q(\xi)$ is the travelling-wave solution, then $\varphi(\xi) = q'(\xi)$

[4] A conserved functional associated with the translation symmetry; see Section 8

[5] Our notation in this example is ambiguous since we denoted the diffusion matrix $D$ of the reaction-diffusion system and the Evans function $D(\lambda)$ by the same letter

Changing variables according to

$$\zeta = \sqrt{|\lambda|}\xi, \qquad \tilde{U} = U, \qquad \sqrt{|\lambda|}\tilde{V} = V,$$

equation (4.4) becomes

$$\begin{pmatrix} \tilde{U}_\zeta \\ \tilde{V}_\zeta \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\mathrm{e}^{\mathrm{i}\arg\lambda} - |\lambda|^{-1}\partial_U F(Q(\zeta/\sqrt{|\lambda|}))) & -c\sqrt{|\lambda|}^{-1} D^{-1} \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}. \tag{4.5}$$

In the limit as $|\lambda| \to \infty$, we obtain the constant-coefficient equation

$$\begin{pmatrix} \tilde{U}_\zeta \\ \tilde{V}_\zeta \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{e}^{\mathrm{i}\arg\lambda} D^{-1} & 0 \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}. \tag{4.6}$$

The Evans function $\tilde{D}(\lambda)$ for the rescaled equation (4.6) can be computed since there is no dependence on $\xi$. We focus on the case $\lambda \in \mathbb{R}^+$, so that $\arg\lambda = 0$: If the diffusion matrix is given by $D = \mathrm{diag}(d_j)$, then the eigenvalues of the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ D^{-1} & 0 \end{pmatrix}$$

are given by $\nu_j = \sqrt{d_j}$ and $\nu_{j+N} = -\sqrt{d_j}$ for $j = 1, \ldots, N$, and the associated eigenvectors are

$$\tilde{u}_j = (e_j, \sqrt{d_j}^{-1} e_j), \qquad \tilde{u}_{j+N} = (-e_j, \sqrt{d_j}^{-1} e_j), \qquad j = 1, \ldots, N$$

where $e_j$ denote the canonical basis vectors in $\mathbb{R}^N$. In particular,

$$\tilde{D}(\lambda) = \det[\tilde{u}_1, \ldots, \tilde{u}_{2N}] = \det \begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ D^{-\frac{1}{2}} & D^{-\frac{1}{2}} \end{pmatrix} = \det[2D^{-\frac{1}{2}}] > 0$$

for $\lambda > 0$. Since the coefficient matrices in equations (4.5) and (4.6) are close to each other, uniformly in $\zeta$, for $\lambda$ sufficiently large, their Evans functions are close uniformly in $\lambda$ as a consequence of Theorem 3.1. In particular, the Evans function $D(\lambda)$ of (4.4) never vanishes for all real, sufficiently large $\lambda$. ■

For the parity argument mentioned above, we would need to compare the Evans function $D_1(\lambda)$ as used in Section 4.2.1 to the Evans function $D_2(\lambda)$ used in Example 1 above. It is not difficult to see that

$$\mathrm{sign} D_1(\lambda) = \mathrm{sign} D_2(\lambda)$$

for all large real $\lambda$ provided the ordered bases $[u_1(0), \ldots, u_N(0)]$ and $[\tilde{u}_1, \ldots, \tilde{u}_N]$ of $\mathrm{N}(P_-(0;0))$ have the same orientation, as do the ordered bases $[u_{N+1}(0), \ldots, u_{2N}(0)]$ and $[\tilde{u}_{N+1}, \ldots, \tilde{u}_{2N}]$ of $\mathrm{R}(P_+(0;0))$. Note that, in set-up of Example 1, we have $k = N$ since we assumed that the essential spectrum is contained in the left half-plane.

## 4.3 Extension across the essential spectrum

We defined the Evans function $D(\lambda)$ for $\lambda$ not in the essential spectrum $\Sigma_{\mathrm{ess}}$ since we were interested in locating the point spectrum. If the essential spectrum is contained in the open left half-plane, then knowing the Evans function for $\lambda$ to the right of the essential spectrum, i.e. in the closed right half-plane, is all we need to decide upon stability. For two important classes of PDEs, however, the essential spectrum will always touch the imaginary axis: these are conservation laws on the one hand and integrable PDEs such as the nonlinear Schrödinger equation

$$\mathrm{i}U_t + U_{xx} + |U|^2 U = 0, \qquad U \in \mathbb{C}$$

and the Korteweg–de Vries equation

$$U_t + U_{xxx} + UU_x = 0, \qquad U \in \mathbb{R}$$

on the other hand.

Thus, suppose that we study the stability of a pulse whose essential spectrum lies on the imaginary axis as is the case for the NLS and the KdV equation. Suppose further that we obtained spectral stability of the pulse, so that the spectrum lies in the closed left half-plane. Assume then that the PDE is perturbed in such a fashion that the pulse under consideration persists as a pulse for the perturbed PDE[6]. The issue is then the stability of the pulse to the perturbed PDE. The essential spectrum can again be calculated easily (see Section 3.4), while perturbations of isolated eigenvalues of the unperturbed PDE can be investigated using Liapunov–Schmidt reduction or the Evans function (see Section 4.2). There is, however, an additional mechanism that can create an instability: Recall that we assumed that the essential spectrum resides on the imaginary axis. Upon adding the perturbation, eigenvalues may bifurcate from the essential spectrum leading to additional point spectrum close to the essential spectrum (see Figure 5c). These eigenvalues are not regular perturbations of the point spectrum of the unperturbed pulse, but are created in the essential spectrum. Since the new eigenvalues bifurcate from the essential spectrum, it is not possible to use Liapunov–Schmidt reduction or standard perturbation theory as Fredholm properties fail. The Evans function, however, can be used to locate and track such eigenvalues as we shall see below.

Consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A(\xi; \lambda) u \tag{4.7}$$

and assume that the limit

$$\lim_{|\xi| \to \infty} A(\xi; \lambda) = A_0(\lambda)$$

exists. On account of the results stated in Section 3.4.4, we have that $\lambda$ is in the essential spectrum of $\mathcal{T}$ precisely when $A_0(\lambda)$ has at least one spatial eigenvalue on the imaginary axis (see Figure 5). If $A_0(\lambda)$ is hyperbolic, then (4.7) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ with projections $P_+(\xi; \lambda)$ and $P_-(\xi; \lambda)$, respectively. An element $\lambda \in \mathbb{C}$ is in the point spectrum precisely if

$$\mathrm{N}(P_-(0; \lambda)) \cap \mathrm{R}(P_+(0; \lambda)) \neq \{0\}.$$

The Evans function $D(\lambda)$ measures intersections of the two subspaces $\mathrm{N}(P_-(0; \lambda))$ and $\mathrm{R}(P_+(0; \lambda))$. It is defined for any $\lambda$ for which $A_0(\lambda)$ is hyperbolic. The key idea is to find an analytic extension of the Evans function $D(\lambda)$ for $\lambda \in \Sigma_{\mathrm{ess}}$. Zeros of the extended function $D(\lambda)$ for $\lambda \in \Sigma_{\mathrm{ess}}$ correspond to possible bifurcation points of point spectrum: Upon adding perturbations, these zeros may move out of the essential spectrum (see Figure 5c).

We illustrate the analytic extension of the Evans function in Figure 5. Throughout this section, we assume that $\lambda$ is close to the essential spectrum $\Sigma_{\mathrm{ess}}$. References for analytic extensions of the Evans function are [134, 72, 97].

First, consider the set-up shown in Figure 5a. When $\lambda$ crosses through $\Sigma_{\mathrm{ess}}$ from right to left, a spatial eigenvalue $\nu$ of $A_0(\lambda)$ crosses through $\mathrm{i}\mathbb{R}$ from right to left. For $\lambda$ to the right of $\Sigma_{\mathrm{ess}}$, the spectral projection $P_0^{\mathrm{s}}(\lambda)$ of the matrix $A_0(\lambda)$ that projects onto the stable eigenspace along the unstable eigenspace is well defined (see (3.4)-(3.5)). Note that stable and unstable spatial eigenvalues for $\lambda$ to the right of $\Sigma_{\mathrm{ess}}$ correspond to the *bullets* and *crosses*, respectively, in Figure 5a. For $\lambda$ on and to the left of $\Sigma_{\mathrm{ess}}$, we denote by $P_0^{\mathrm{s}}(\lambda)$ the spectral projection of the matrix $A_0(\lambda)$ associated with the two spectral sets consisting of the bullets and

[6]We refer to Section 2.2 for examples of such perturbations for the NLS and the KdV equation

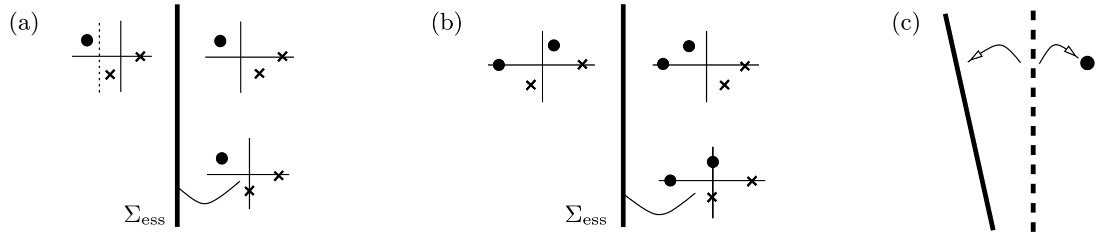


**Figure 5** Plotted is the complex $\lambda$-plane. The insets represent the spatial spectra of the matrix $A_0(\lambda)$ in different regions of the $\lambda$-plane. Two different spatial eigenvalue configurations near the essential spectrum are plotted: In (a), a single spatial eigenvalue crosses the imaginary axis when $\lambda$ crosses through $\Sigma_{\text{ess}}$; the dotted line in the inset is given by $\operatorname{Re}\nu = \eta$. In (b) two spatial eigenvalues cross simultaneously but in opposite directions when $\lambda$ crosses through $\Sigma_{\text{ess}}$. In (c), we plotted the unperturbed essential spectrum (dashed line) as well as the perturbed essential spectrum (solid line) and an additional eigenvalue that moves out of the essential spectrum upon perturbing the operator.

the crosses, respectively, in Figure 5a. The projection is well defined as long as $\lambda$ is close to $\Sigma_{\text{ess}}$ and as long as spatial eigenvalues cross only from right to left through $\mathrm{i}\mathbb{R}$ as $\lambda$ crosses from right to left through $\Sigma_{\text{ess}}$. In other words, rather than dividing $\operatorname{spec}(A_0(\lambda))$ into stable and unstable spatial eigenvalues, we distinguish between spatial eigenvalues $\nu$ with $\operatorname{Re}\nu < \eta$ and $\operatorname{Re}\nu > \eta$ where $\eta < 0$ is close to zero such that the line $\operatorname{Re}\nu = \eta$ separates the stable spatial eigenvalues from the formerly unstable spatial eigenvalues that crossed the imaginary axis (see Figure 5a). The discussion at the end of Section 3.2 shows that there are projections $P_+(\xi;\lambda)$ and $P_-(\xi;\lambda)$, defined for $\xi \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^-$ respectively, that are defined and analytic in $\lambda$ for all $\lambda$ close to $\Sigma_{\text{ess}}$ such that

$$P_\pm(\xi;\lambda) \longrightarrow P_0^{\text{s}}(\lambda), \qquad \xi \to \pm\infty.$$

The subspace $\mathrm{R}(P_+(0;\lambda))$ consists of precisely those initial conditions that lead to solutions $u(\xi)$ of (4.7) that satisfy $|u(\xi)| \le \mathrm{e}^{\eta\xi}|u(0)|$ for $\xi \ge 0$. The subspace $\mathrm{N}(P_-(0;\lambda))$ consists of precisely those initial conditions $u(0)$ for which $|u(\xi)| \le \mathrm{e}^{\eta\xi}|u(0)|$ for $\xi \le 0$; note that these solutions will in general grow exponentially as $\xi \to -\infty$ whenever $\lambda$ is to the left of $\Sigma_{\text{ess}}$.

The Evans function $D(\lambda)$ can now be defined as in Section 4.1 via $D(\lambda) = \mathrm{N}(P_-(0;\lambda)) \wedge \mathrm{R}(P_+(0;\lambda))$, so that zeros of $D(\lambda)$ correspond to non-trivial intersections of $\mathrm{R}(P_+(0;\lambda))$ and $\mathrm{N}(P_-(0;\lambda))$. Note that zeros of $D(\lambda)$ for $\lambda$ to the right of $\Sigma_{\text{ess}}$ correspond to eigenvalues of $\mathcal{T}$, but that zeros to the left or on $\Sigma_{\text{ess}}$ have no meaning for $\mathcal{T}$. The latter zeros are commonly referred to as resonance poles.

We conclude that eigenvalues can bifurcate from the essential spectrum only at zeros $\lambda_* \in \Sigma_{\text{ess}}$ of the extended Evans function $D(\lambda)$. Suppose that this extension is computed, and its zeros determined. Upon adding perturbations to (4.7) that depend on the small perturbation parameter $\epsilon$, a perturbation analysis of the analytic function $D(\lambda;\epsilon)$ near each zero then reveals whether the zeros move to the right or to the left of $\Sigma_{\text{ess}}$. This program has been carried out for the generalized Korteweg–de Vries equation

$$U_t + U_{\xi\xi\xi} - cU_\xi + U^p U_\xi = 0, \qquad U \in \mathbb{R}$$

by Pego and Weinstein [134, 135]: they showed that pulses destabilize at $p = 4$ where an eigenvalue emerges from the essential spectrum, given by the imaginary axis $\mathrm{i}\mathbb{R}$, at $\lambda = 0$.

An alternative, but equivalent, way of extending the Evans function in the situation shown in Figure 5a consists of using the transformation (3.9)

$$v(\xi) = u(\xi)\mathrm{e}^{-\eta\xi}$$

which replaces (4.7) by

$$\frac{\mathrm{d}}{\mathrm{d}\xi} v = (A(\xi;\lambda) - \eta)v. \tag{4.8}$$

For $\eta < 0$ close to zero, the asymptotic matrix $A(\lambda) - \eta$ has imaginary eigenvalues for values of $\lambda$ on a curve $\Sigma_{\rm ess}^{\eta}$ that is strictly to the left of $\Sigma_{\rm ess}$ (see Figure 5a). Thus, the Evans function for (4.8) is defined for all $\lambda$ near $\Sigma_{\rm ess}$ and coincides with the Evans function of (4.7) for $\lambda$ to the right of $\Sigma_{\rm ess}$.

Next, we consider the case illustrated in Figure 5b. In this situation, two spatial eigenvalues cross the imaginary axis simultaneously in opposite directions as $\lambda$ crosses through $\Sigma_{\rm ess}$. We are interested in extending the Evans function, defined in the region to the right of $\Sigma_{\rm ess}$, in an analytic fashion to the left of $\Sigma_{\rm ess}$. The procedure outlined above for case (a) seems to fail since there is no spectral gap anymore. Recall that the Evans function $D(\lambda)$ for $\lambda$ to the right of $\Sigma_{\rm ess}$ is defined by

$$D(\lambda) = \det[u_1(\lambda), \dots, u_n(\lambda)],$$

where $[u_1(\lambda), \dots, u_k(\lambda)]$ and $[u_{k+1}(\lambda), \dots, u_n(\lambda)]$ are ordered bases of $\mathrm{N}(P_-(0;\lambda))$ and $\mathrm{R}(P_+(0;\lambda))$. One way of obtaining an analytic extension of the Evans function in this set-up is to analytically extend the exponential dichotomies $\Phi_+^{\rm s}(\xi, 0)$ for $\xi \geq 0$ and $\Phi_-^{\rm u}(\xi, 0)$ for $\xi \leq 0$; recall that $\Phi_+^{\rm s}(0,0) = P_+(0;\lambda)$ and $\Phi_-^{\rm u}(0,0) = \mathrm{id} - P_-(0;\lambda)$. For simplicity, we concentrate on $\xi \geq 0$.

We begin by extending the projections of the asymptotic constant-coefficient equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = A_0(\lambda) u. \tag{4.9}$$

The stable and unstable spatial eigenvalues of $A_0(\lambda)$ for $\lambda$ to the right of $\Sigma_{\rm ess}$ correspond to the *bullets* and *crosses*, respectively, in Figure 5b. For $\lambda$ on and to the left of $\Sigma_{\rm ess}$, we denote by $P_0^{\rm s}(\lambda)$ the spectral projection of the matrix $A_0(\lambda)$ onto the generalized eigenspace associated with the bullets along the eigenspace associated with the crosses. In other words, we assume that we can divide the spatial spectrum of $A_0(\lambda)$ into two disjoint spectral sets according to whether the spatial eigenvalues are in the left or right half-plane for $\lambda$ to the right of $\Sigma_{\rm ess}$; thus, we do not allow that formerly stable and unstable spatial eigenvalues cross through $\mathrm{i}\mathbb{R}$ at the same point $\mathrm{i}k$.[7] Using Dunford's integral [100, Chapter I.5.6], we obtain spectral projections, which we still denote by $P_0^{\rm s}(\lambda)$, that depend analytically on $\lambda$ for $\lambda$ to the left of the essential spectrum $\Sigma_{\rm ess}$. The evolution operators of (4.9) are

$$\mathrm{e}^{A_0(\lambda)(\xi-\zeta)} P_0^{\rm s}(\lambda) \quad \text{and} \quad \mathrm{e}^{A_0(\lambda)(\xi-\zeta)} P_0^{\rm u}(\lambda) \tag{4.10}$$

for $\lambda$ in a, possibly small, neighbourhood of $\Sigma_{\rm ess}$, where $P_0^{\rm u}(\lambda) = \mathrm{id} - P_0^{\rm s}(\lambda)$. In the next step, we construct the evolution operator $\Phi^{\rm s}(\xi, 0)$ for the full equation (4.7). Thus, assume that $\Phi^{\rm s}(\xi, \zeta)$ is associated with (4.7) on $\mathbb{R}^+$ for $\lambda$ to the right of the essential spectrum. Roughly speaking, analyticity of these operators with respect to $\lambda$ is equivalent to uniqueness. Thus, we shall find a way to extend these operators uniquely to the region to the left of $\Sigma_{\rm ess}$. The idea is to seek an extension of these dichotomies by requiring that the extended evolution operators approximate the dichotomies (4.10) of the asymptotic constant-coefficient equation (4.9) as best as possible. Intuitively, such a choice should be possible provided the coefficient matrix $A(\xi;\lambda)$ converges rapidly enough to the asymptotic coefficient matrix $A_0(\lambda)$. In other words, the solutions of (4.7) should converge rapidly towards solutions of (4.9) provided the coefficient matrix $A(\xi;\lambda)$ converges rapidly to $A_0(\lambda)$. This is indeed the case as long as the distance, in the real part, between the rightmost formerly stable spatial eigenvalue and the leftmost formerly unstable eigenvalue, i.e. the amount of overlap of the two spectral sets, is smaller than $\rho$, where $\rho$ is the exponential rate with which the coefficient matrices approach each other (see (4.11) below). This result is referred to as the Gap Lemma [72, 97]. To illustrate this result, we therefore assume that

$$|A(\xi;\lambda) - A_0(\lambda)| \leq C\mathrm{e}^{-\rho|\xi|} \tag{4.11}$$

[7] This assumption is actually not necessary, and we refer to [97] for the details

as $|\xi| \to \infty$ for some $\rho > 0$ independently of $\lambda$. To construct the extension of $\Phi^s(\xi, 0)$, we define the correction $\Psi^s(\xi, 0)$ by

$$\Phi^s(\xi, 0) = e^{A_0(\lambda)\xi} P_0^s(\lambda) + \Psi^s(\xi, 0)$$

and seek the correction $\Psi^s(\xi, 0)$ as a solution to the integral equation

$$\begin{aligned} \Psi^s(\xi, 0) \quad = \quad & \int_\infty^\xi e^{A_0(\lambda)(\xi-\tau)} P_0^u(\lambda)\Delta(\tau; \lambda)[e^{A_0(\lambda)\tau} P_0^s(\lambda) + \Psi^s(\tau, 0)]\, d\tau \\ & + \int_0^\xi e^{A_0(\lambda)(\xi-\tau)} P_0^s(\lambda)\Delta(\tau; \lambda)[e^{A_0(\lambda)\tau} P_0^s(\lambda) + \Psi^s(\tau, 0)]\, d\tau, \qquad \xi \geq 0 \end{aligned} \tag{4.12}$$

where

$$\Delta(\xi; \lambda) = A(\xi; \lambda) - A_0(\lambda) = O(e^{-\rho|\xi|}).$$

The above integral equation coincides with the integral equation (3.8), which we encountered in Section 3.2 when we constructed regular exponential dichotomies, upon substituting the above expression for $\Phi^s$ and setting $\zeta = 0$ in (3.8). Equation (4.12) is also reminiscent of the integral equation that describes strong stable manifolds; it has a unique solution (possibly after replacing $\xi = 0$ by $\xi = L$ in the second integral for some $L \gg 1$ to make the right-hand side of (4.12) a contraction) that gives the correction $\Psi^s$. The exponential decay in (4.11) is necessary to compensate for the exponential growth of the solution operators in (4.10).

Hence, we can construct analytic extension of the dichotomies for (4.7), and thus of the Evans function $D(\lambda)$, for $\lambda$ to the left of the essential spectrum. Slightly different constructions have been carried out in [72, 97]. In [72], the analytically extended Evans function has been used to establish instability criteria of shock waves to conservation laws. In [97, 98, 96, 110], this approach was used to prove stability and instability of solitary waves to perturbations of the nonlinear Schrödinger equation. We also refer to [45, 46, 47] for applications, and extensions, of the analytic extension of the Evans function across the essential spectrum to the stability of pulses in singularly perturbed reaction-diffusion equations with a strong coupling between the fast and slow subsystems.

# 5 Spectral stability of multi-bump pulses

In this section, we consider the stability of multi-bump pulses. Suppose that we know that a given PDE supports a stable pulse $Q(x - c_0 t)$ that travels with speed $c_0$. We refer to this pulse as the primary pulse. Typically, such a pulse is then accompanied by spatially-periodic wave trains $P_L(x - c_L t)$ that resemble infinitely-many equidistant copies of the pulse (see Figure 6a-b). These wave trains have spatial period $2L$ and wave speed $c_L$; they exist for any sufficiently large spatial period $L$, and the wave speeds satisfy $c_L \to c_0$ as $L \to \infty$. Besides these long-wavelength pulse trains, multi-bump pulses may exist which are travelling waves that consist of several well-separated copies of the primary pulse. Associated with an $\ell$-pulse, consisting of $\ell$ copies of the primary pulse, are the distances $2L_1, \dots, 2L_{\ell-1}$ between consecutive copies and the locations $\xi_1, \dots, \xi_\ell$ of the individual pulses (see Figure 6c for a plot of a 3-pulse). Throughout the entire section, we assume that consecutive pulses in a wave train or an $\ell$-pulse are well separated, so that $L$ and $L_j$ are sufficiently large, say larger than some number $L_* \gg 1$.

We are interested in the spectra of the wave trains and the $\ell$-pulses (if they exist) that accompany the primary pulse. We assume that the primary pulse is spectrally stable, so that its spectrum is contained in the open left half-plane with the exception of a simple eigenvalue at $\lambda = 0$, caused by the translation symmetry (see Figure 7a). We shall locate the spectrum of an $\ell$-pulse that accompanies the primary pulse: The essential spectrum of an $\ell$-pulse is close to the essential spectrum of the primary pulse as it is determined by the asymptotic rest state (see Section 3.4.4). It remains to find the point spectrum. We claim that there are

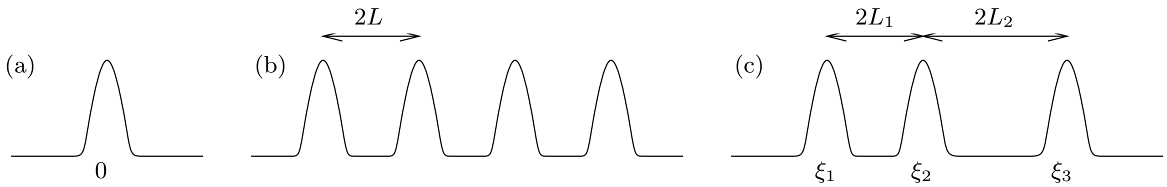


**Figure 6** Plots of the primary pulse $Q(\xi)$ in (a), a spatially-periodic wave train with period $2L$ in (b), and a 3-pulse with distances $2L_1$ and $2L_2$ in (c).

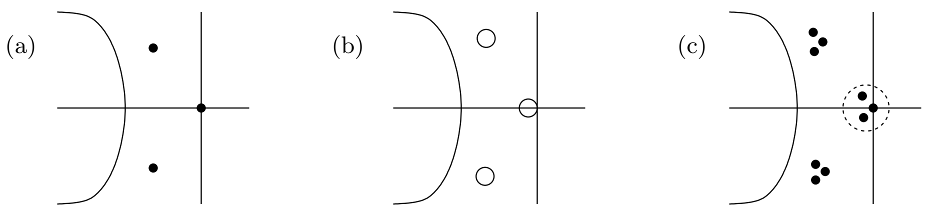


**Figure 7** Plots of the spectra of the primary pulse in (a), a spatially-periodic wave train in (b), and a 3-pulse in (c).

precisely $\ell$ eigenvalues in the spectrum of an $\ell$-pulse near each eigenvalue of the primary pulse (see Figure 7c). Recall that, as usual, all eigenvalues are counted with their multiplicity. To make our claim plausible, we argue heuristically and focus on the eigenvalues near $\lambda = 0$. Consider the 3-pulse plotted in Figure 6c. Each individual pulse in the 3-pulse resembles the primary pulse, and the individual pulses are well separated. Thus, if we change the position of one of the pulses, the other pulses are not affected. Translating the $j$th pulse corresponds to adding $\epsilon Q'(\cdot - \xi_j)$ to the 3-pulse for $\epsilon$ small. Hence, there should be three eigenvalues near $\lambda = 0$, and the associated eigenfunctions should be linear combinations of $Q'(\cdot - \xi_j)$ for $j = 1, 2, 3$, since adding small multiples of these eigenfunctions to the 3-pulse does not affect the 3-pulse much. The key to this argument is that the pulses are exponentially localized and well separated. Thus, an $\ell$-pulse should have $\ell$ eigenvalues near $\lambda = 0$ with eigenfunctions of the form $\sum_{j=1}^{\ell} d_j Q'(\cdot - \xi_j)$. The larger the distances between consecutive pulses in the $\ell$-pulse, the closer should these $\ell$ critical eigenvalues be to $\lambda = 0$. If the primary pulse is stable, then an $\ell$-pulse is stable provided its $\ell - 1$ non-trivial critical eigenvalues near $\lambda = 0$ move into the left half-plane.

Similar arguments apply to the wave trains. Since a wave train consists of infinitely many pulses, there should be many eigenvalues near $\lambda = 0$. In fact, for each eigenvalue of the pulse, the wave train has a circle of eigenvalues that is parametrized by the spatial Floquet exponent $\gamma$ (see Section 3.4.2, in particular (3.16), and Figure 7b).

To set-up the problem, consider the travelling-wave ODE

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = f(u, c) \tag{5.1}$$

where $c$ denotes the wave speed. We assume that the rest state is $u = 0$, so that $f(0, c) = 0$ for all $c$. Suppose that $q(\xi)$ is a pulse to $u = 0$ for $c = c_0$. We assume that $\partial_u f(0, c_0)$ is hyperbolic, so that $\rho < \min\{|\operatorname{Re} \nu|;\ \nu \in \operatorname{spec}(\partial_u f(0, c_0))\} < 3\rho/2$ for an appropriate $\rho > 0$; it is advantageous to choose $\rho$ as large as possible as it appears in the estimates for certain remainder terms (see below).

We remark that, if the travelling waves are, in fact, standing waves so that $c = 0$, then the underlying PDE sometimes features the reflection symmetry $x \mapsto -x$ that manifest itself as a so-called reversibility of the travelling-wave ODE (5.1). We would then be interested in symmetric waves that are invariant under the reflection $x \mapsto -x$. We refer to [174, 181] for more background on reversible ODEs.

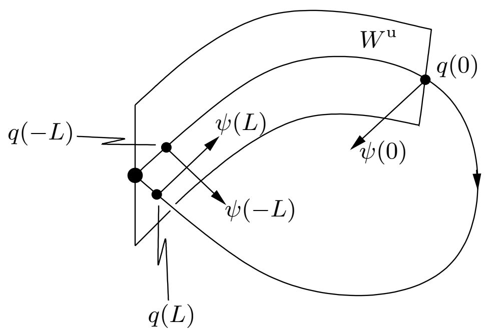


**Figure 8** The geometry of the pulse $q(\xi)$ and the solution $\psi(\xi)$ to the adjoint variational equation. Note that $\psi(\xi) \perp (T_{q(\xi)}W^{\mathrm{s}}(0) + T_{q(\xi)}W^{\mathrm{u}}(0))$ for all $\xi$ (see Remark 3.5), i.e. $\psi(\xi)$ is perpendicular to the tangent spaces of both the stable and the unstable manifold[8] of the equilibrium that is approached by the pulse.

The PDE eigenvalue problem associated with the pulse $q(\xi)$ is of the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = [\partial_u f(q(\xi), c_0) + \lambda B(\xi)]u.$$

We make the following assumptions.

**Hypothesis 5.1** *The only bounded solution to the variational equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = \partial_u f(q(\xi), c_0)u \tag{5.2}$$

*is given by $q'(\xi)$, up to constant scalar multiples.*

As a consequence, the adjoint variational equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = -\partial_u f(q(\xi), c_0)^* u \tag{5.3}$$

has a unique, up to constant multiples, bounded non-zero solution which we denote by $\psi(\xi)$ (see Section 3.3). We refer to Figure 8 for the geometry of the pulse $q(\xi)$ and the solution $\psi(\xi)$.

**Hypothesis 5.2** *We assume that*

$$M = \int_{-\infty}^{\infty} \langle \psi(\xi), Bq'(\xi) \rangle \, \mathrm{d}\xi = -\int_{-\infty}^{\infty} \langle \psi(\xi), \partial_c f(q(\xi), c_0) \rangle \, \mathrm{d}\xi \neq 0$$

*is not zero (recall (2.8)).*

In the notation used earlier, the above hypotheses say that $0 \in \Sigma_{\mathrm{pt}}$, $\mathrm{N}(\mathcal{T}(0)) = \mathrm{span}\{q'\}$ and $D'(0) \neq 0$, i.e. that $\lambda = 0$ is a simple eigenvalue of $\mathcal{T}$ (see Sections 3 and 4). In other words, we assume that $\lambda = 0$ is an isolated simple eigenvalue of the pulse $q(\xi)$. The next hypothesis is not always needed.

**Hypothesis 5.3** *We assume that the point spectrum $\Sigma_{\mathrm{pt}}$ of the primary pulse is a discrete subset of $\mathbb{C}$.*

[8] The stable manifold $W^{\mathrm{s}}(p_0)$ of an equilibrium $u = p_0$ consists of all solutions that converge to that equilibrium as $\xi \to \infty$; analogously, its unstable manifold $W^{\mathrm{u}}(p_0)$ consists of all solutions that converge to $p_0$ as $\xi \to -\infty$ (see [13, 33, 80, 107])

We are interested in the stability of $2L$-periodic wave trains $p_L(\xi)$ and $\ell$-pulses $q_\ell(\xi)$ that accompany the primary pulse $q(\xi)$.

Throughout this section, we assume that, for a sufficiently small constant $\delta > 0$, we have

$$|c - c_0| < \delta, \qquad u(\xi) \in \mathcal{U}_\delta\Big(\overline{\{q(\zeta);\ \zeta \in \mathbb{R}\}}\Big) \qquad \text{for all } \xi \in \mathbb{R} \tag{5.4}$$

for any wave $u(\xi)$ with wave speed $c$ that we consider below. In particular, we assume that the period $2L$ of any wave train we may consider is sufficiently large. Furthermore, we denote by $\Sigma(u) = \Sigma_{\mathrm{pt}}(u) \cup \Sigma_{\mathrm{ess}}(u)$ the various spectra of a wave $u(\xi)$ with speed $c$, computed with respect to the operator

$$\mathcal{T}_u(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - \partial_u f(u(\cdot), c) - \lambda B(\cdot).$$

## 5.1 Spatially-periodic wave trains with long wavelength

Suppose that $p_L(\xi)$ is a $2L$-periodic wave-train solution of (5.1) with wave speed $c_L$, so that $p_L(-L) = p_L(L)$. We comment later on the existence of such wave trains (see [11, 111, 181]). We recall from Section 3.4.2 that $\lambda$ is in the spectrum of $p_L(\xi)$ if, and only if, the boundary-value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} u &= [\partial_u f(p_L(\xi), c_L) + \lambda B(\xi)] u, \qquad |\xi| < L \\ u(L) &= \mathrm{e}^{\mathrm{i}\gamma} u(-L) \end{aligned} \tag{5.5}$$

has a solution $u(\xi)$ for some $\gamma \in \mathbb{R}$.

**Theorem 5.1 ([69])** *Assume that Hypothesis 5.3 is met. For every sufficiently small $\epsilon > 0$, a $\delta > 0$ exists with the following properties. If $p_L(\xi)$ is a $2L$-periodic wave-train solution of (5.1) such that (5.4) is met, then the following statements are true.*

- $\Sigma(p_L) = \Sigma_{\mathrm{ess}}(p_L) \subset \mathcal{U}_\epsilon(\Sigma(q))$.
- *For any $\lambda_* \in \Sigma_{\mathrm{pt}}(q)$ with multiplicity $m$, and for any fixed spatial Floquet exponent $\gamma \in [0, 2\pi)$, (5.5) has precisely $m$ solutions, counted with multiplicity, in the $\epsilon$-neighbourhood of $\lambda_*$.*

Besides the topological proof given in [69], Theorem 5.5 can also be proved using the roughness theorem of exponential dichotomies; we refer to [155, Section 4].

**Remark 5.1** *It follows from the results in [155, Section 5.2] that the essential spectrum $\Sigma_{\mathrm{ess}}(q)$ is also approximated by the spectrum $\Sigma(p_L)$ of the wave trains.*

Thus, if the spectrum $\Sigma(q)$ of the primary pulse is contained in the open left half-plane with the exception of a simple eigenvalue at $\lambda = 0$, then the spectrum $\Sigma(p_L)$ of the wave train is contained in the open left half-plane with the exception of a circle $\lambda(\gamma)$ of simple eigenvalues that is parametrized by $\gamma \in [0, 2\pi]$ with $\lambda(0) = 0$. To conclude (in)stability of the wave train, it is necessary to locate this circle of critical eigenvalues.

**Theorem 5.2 ([156])** *Assume that Hypotheses 5.1 and 5.2 are met. There is a $\delta > 0$ with the following properties. Assume that $p_L(\xi)$ is a $2L$-periodic wave-train solution of (5.1) such that (5.4) is met. Equation (5.5) has a solution for $\gamma \in \mathbb{R}$ and $\lambda$ close to zero if, and only if,*

$$\lambda = \frac{1}{M}\Big((\mathrm{e}^{\mathrm{i}\gamma} - 1)\langle\psi(L), q'(-L)\rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma})\langle\psi(-L), q'(L)\rangle\Big) + R(\gamma, L), \tag{5.6}$$

*where $R(\gamma, L)$ is of the form*

$$R(\gamma, L) = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\mathrm{O}(\mathrm{e}^{-3\rho L}) + (1 - \mathrm{e}^{-\mathrm{i}\gamma})\mathrm{O}(\mathrm{e}^{-3\rho L}).$$

*The associated solution $u(\xi)$ of (5.5) is given by*

$$u(\xi) = \mathrm{e}^{\mathrm{i}k\gamma} q'(\xi) + \mathrm{O}(\mathrm{e}^{-\rho L}), \qquad \xi \in [(2k-1)L, (2k+1)L], \quad k \in \mathbb{Z}.$$

*Note that the sign of*

$$\frac{\mathrm{d}^2}{\mathrm{d}\gamma^2} \operatorname{Re} \lambda \Big|_{\gamma=0} = \frac{1}{M}\Big( - \langle \psi(L), q'(-L)\rangle + \langle \psi(-L), q'(L)\rangle \Big)$$

*decides, to leading order, upon stability.*

Reference [156] contains more general results, with better estimates for $R(\gamma, L)$, that are applicable even if $\lambda$ has larger geometric multiplicity due to the presence of additional continuous symmetries.

Before we illustrate the theorem by examples, we outline its proof as it provides some insight as to the role of the scalar products appearing in (5.6) (see also Figure 8).

### 5.1.1 Outline of the proof of Theorem 5.2

Since the ideas for the proof of the stability theorem 5.2 are the same as those that give existence of wave trains, we begin by introducing Lin's method [111, 143] that can be used to prove existence. Thus, to set the scene, suppose we want to prove the existence of the periodic orbits $p_L(\xi)$ of period $2L$ near a given homoclinic orbit $q(\xi)$ to the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u = f(u, c), \qquad u \in \mathbb{R}^2$$

in the plane, where $c$ denotes the wave speed. Hence, we shall find functions $p_L(\xi)$ and wave speeds $c_L$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} p_L(\xi) &= f(p_L(\xi), c_L), \qquad |\xi| < L \\ p_L(0) &\in q(0) + \operatorname{span}\{\psi(0)\} = \Delta \\ p_L(-L) &= p_L(L) \end{aligned}$$

and such that $c_L$ is close to $c_0$ and $p_L(\xi)$ is close to $q(\xi)$ for $|\xi| < L$. Note that $\Delta$ is a line attached to $q(0)$ and perpendicular to the vector field (see Figure 9a). To solve this problem, we shall seek functions

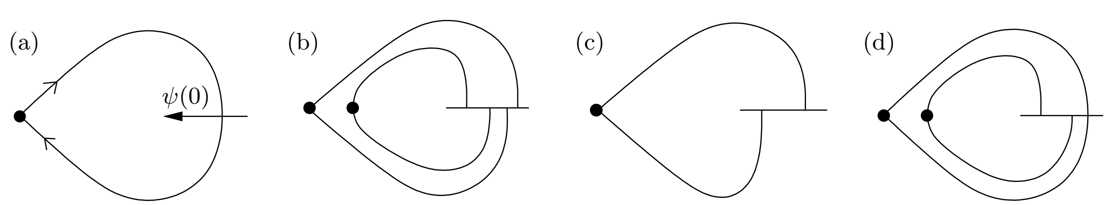


**Figure 9** The homoclinic orbit $q(\xi)$ and the transverse section $\Delta = q(0) + \operatorname{span}\{\psi(0)\}$ that is attached to $q(0)$ are plotted in (a). In (b), the solution $u(\xi, c, L)$ to (5.7) is plotted for $c \neq c_0$. In (c), the perturbed stable and unstable manifolds, denoted by $u(\xi; c, \infty)$ are plotted for $c \neq c_0$; the distance between the unstable and the stable manifold is, to leading order, given by $u(0-; c, \infty) - u(0+; c, \infty) = -M(c - c_0)$. Lastly, in (d), we plotted the unique solution $u(\xi; c_0, L)$ to (5.7) for $c = c_0$.

$u(\xi; c, L)$, defined for all $c$ close to $c_0$ and all $L$ large, that may have a discontinuity[9] at $\xi = 0$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} u(\xi; c, L) &= f(u(\xi; c, L), c), \qquad |\xi| < L \\ u(0\pm; c, L) &\in q(0) + \mathrm{span}\{\psi(0)\} = \Delta \\ u(-L; c, L) &= u(L; c, L) \end{aligned} \tag{5.7}$$

(see Figure 9b). Such a solution is the desired periodic wave train if, and only if,

$$\Xi(c, L) := u(0-; c, L) - u(0+; c, L) = 0$$

so that $u(\xi; c, L)$ is continuous at $\xi = 0$. Hence, we focus on solving (5.7). First, we pretend that we can expand the solution $u(\xi; c, L)$ as a Taylor series in $(c, L)$ centered at $(c, L) = (c_0, \infty)$. In other words, we write

$$u(\xi; c, L) = u(\xi; c, \infty) + u(\xi; c_0, L) \tag{5.8}$$

where $u(\xi; c, \infty)$ satisfies (5.7) for $L = \infty$ (see Figure 9c), while $u(\xi; c_0, L)$ satisfies (5.7) for $c = c_0$ (see Figure 9d). Note that the solution $u(\xi; c, \infty)$ of (5.7) with $L = \infty$ are precisely the stable and unstable manifolds: in other words, $u(\xi; c, \infty)$ parametrizes the stable manifold for $\xi \geq 0$ and the unstable manifold for $\xi \leq 0$ with a discontinuity at $\xi = 0$ (see Figure 9c). Owing to Melnikov theory [111], the jump at $\xi = 0$ is given

$$\begin{aligned} \langle \psi(0), u(0-; c, \infty) - u(0+; c, \infty) \rangle &= (c - c_0) \int_{-\infty}^{\infty} \langle \psi(\xi), \partial_c f(q(\xi), c_0) \rangle \, \mathrm{d}\xi + \mathrm{O}(|c - c_0|^2) \\ &= -M(c - c_0) + \mathrm{O}(|c - c_0|^2). \end{aligned} \tag{5.9}$$

It remains to find $u(\xi; c_0, L)$. Hence, set $c = c_0$ and write

$$u(\xi; c_0, L) = q(\xi) + v(\xi), \tag{5.10}$$

then $u(\xi; c_0, L)$ satisfies the ODE in (5.7) if, and only if, $v(\xi)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\xi} v = f(q(\xi) + v, c_0) - f(q(\xi), c_0) = \partial_u f(q(\xi), c_0) v + \mathrm{O}(|v|^2), \qquad |\xi| < L. \tag{5.11}$$

Omitting the higher-order term $\mathrm{O}(|v|^2)$, the solution to this equation can be written, for instance, as

$$\begin{aligned} v(\xi) &= \Phi(\xi, -L) v(-L) \qquad \xi \geq 0 \\ v(\xi) &= \Phi(\xi, L) v(L) \qquad \xi \leq 0 \end{aligned}$$

where $\Phi(\xi, \zeta)$ is the evolution of the linear part of (5.11). At $\xi = \pm L$, we shall satisfy

$$q(L) + v(L) = q(-L) + v(-L).$$

Thus, we set $v(L) = q(-L)$ and $v(-L) = q(L)$, and the solution $v(\xi)$ to (5.11) is therefore given by

$$v(\xi) = \begin{cases} \Phi(\xi, -L) v(-L) = \Phi(\xi, -L) q(L) & \xi \in [-L, 0] \\ \Phi(\xi, L) v(L) = \Phi(\xi, L) q(-L) & \xi \in [0, L] \end{cases}$$

upon omitting the terms of higher order in (5.11). In particular,

$$v(0-) = \Phi(0, -L) q(L), \qquad v(0+) = \Phi(0, L) q(-L).$$

[9]If $u(\xi)$ has a discontinuity at $\xi = 0$, we define $u(0-) = \lim_{\xi \nearrow 0} u(\xi)$ and $u(0+) = \lim_{\xi \searrow 0} u(\xi)$

In summary, using (5.8) and (5.9), we obtain that the discontinuity $\Xi(c, L)$ of $u(\xi; c, L)$ at $\xi = 0$ is given by

$$\begin{aligned}\Xi(c, L) &= \langle \psi(0), u(0-; c, L) - u(0+; c, L) \rangle \\ &= \langle \psi(0), v(0-) \rangle - \langle \psi(0), v(0+) \rangle + \langle \psi(0), u(0-; c, \infty) - u(0+; c, \infty) \rangle \\ &= \langle \psi(-L), q(L) \rangle - \langle \psi(L), q(-L) \rangle - M(c - c_0)\end{aligned}$$

upon using that

$$\psi(\xi) = \Phi(0, \xi)^* \psi(0)$$

(see Remark 3.4). Thus, a $2L$-periodic wave train with wave speed $c$ exists if, and only if,

$$M(c - c_0) = \langle \psi(-L), q(L) \rangle - \langle \psi(L), q(-L) \rangle.$$

In particular, $2L$-periodic wave trains exist under our assumptions for any $L$ sufficiently large, and the nonlinear dispersion relation that relates wave speed and period is given by

$$c_L = c_0 + \frac{1}{M} \Big( \langle \psi(-L), q(L) \rangle - \langle \psi(L), q(-L) \rangle \Big).$$

We refer to [11, 111, 143, 181, 144, 156] for more details that justify the above heuristic argument.

The stability theorem 5.6 can be proved in a very similar fashion by applying the approach outlined above to the linear equation (5.5) with the wave-speed parameter $c$ replaced by the eigenvalue parameter $\lambda$. Regarding the parametrization (5.10), it is also advantageous to use

$$u(\xi; c_L, L) = p_L'(\xi) + v(\xi)$$

instead of $u(\xi; c_0, L) = q'(\xi) + v(\xi)$: this allows us to exploit that $p_L'(\xi)$ is an exact solution to (5.5) for $\lambda = 0$ and $\gamma = 0$. We refer to [156] for the details.

### 5.1.2 Discussion of Theorem 5.2

Theorem 5.2 gives the location of the circle of critical eigenvalues near $\lambda = 0$:

$$\lambda = \frac{1}{M} \Big( (\mathrm{e}^{\mathrm{i}\gamma} - 1) \langle \psi(L), q'(-L) \rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma}) \langle \psi(-L), q'(L) \rangle \Big) \tag{5.12}$$

where we neglected the remainder term $R(\gamma, L)$. To apply this result, we need to find expressions for the scalar products that involve the solutions $\psi(\xi)$ to the adjoint equation (5.3) and $q'(\xi)$ to the variational equation (5.2). Note, however, that these solutions are needed only for $|\xi|$ large. The tails of these solutions are determined by the eigenvalue structure of the matrix $\partial_u f(0, c_0)$ (see, for instance, [34, Chapter 3.8] or [83, Chapter X.13]). We briefly outline the two situations that occur generically.

First, we assume that the leading eigenvalue[10] of $\partial_u f(0, c_0)$ is real and simple. For simplicity, we also assume that its real part is negative (the analysis for a positive real part is completely analogous and can, in fact, be found in [156]). Thus, there is a simple spatial eigenvalue $\nu^{\mathrm{s}} \in \mathrm{spec}(\partial_u f(0, c_0))$ such that $|\operatorname{Re} \nu| > -\nu^{\mathrm{s}} > 0$ for every $\nu \in \mathrm{spec}(\partial_u f(0, c_0))$ with $\nu \neq \nu^{\mathrm{s}}$ (see Figure 10a). It follows from [34, Chapter 3.8] or [83, Chapter X.13] that there is a $\delta > 0$ and eigenvectors $v_0$ and $w_0$ of $\partial_u f(0, c_0)$ and $\partial_u f(0, c_0)^*$, respectively, belonging to the eigenvalue $\nu^{\mathrm{s}}$ such that

$$q'(\xi) = \mathrm{e}^{\nu^{\mathrm{s}} \xi} v_0 + \mathrm{O}(\mathrm{e}^{-(|\nu^{\mathrm{s}}| + \delta)\xi}), \qquad \psi(-\xi) = \mathrm{e}^{\nu^{\mathrm{s}} \xi} w_0 + \mathrm{O}(\mathrm{e}^{-(|\nu^{\mathrm{s}}| + \delta)\xi})$$

as $\xi \to \infty$. Typically, the vectors $v_0$ and $w_0$ are non-zero. Furthermore, we have

$$|q'(-\xi)| + |\psi(\xi)| = \mathrm{O}(\mathrm{e}^{(|\nu^{\mathrm{s}}| + \delta)\xi})$$

[10] The leading, or principal, eigenvalues of a hyperbolic matrix are those closest to the imaginary axis

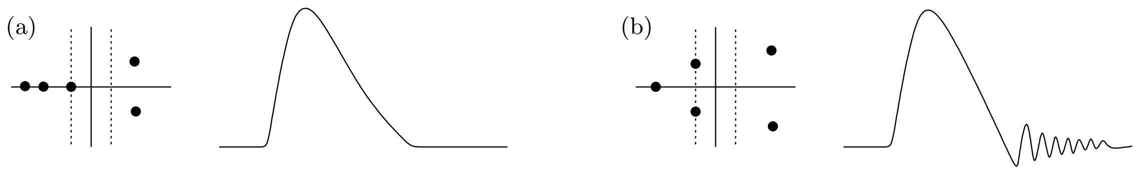


**Figure 10** The spectrum of $\partial_u f(0, c_0)$ and the shape of the tails of the pulse $q(\xi)$ are plotted for two different cases: in (a), the leading eigenvalue $\nu^{\mathrm{s}}$ is simple and real with negative real part, whereas the leading eigenvalues in (b) are a pair of simple, complex-conjugate eigenvalues with negative real part.

as $\xi \to \infty$. Upon substituting these expressions and estimates for $q'$ and $\psi$ into (5.12), we obtain

$$\lambda = \frac{\langle v_0, w_0 \rangle}{M} (1 - \mathrm{e}^{-\mathrm{i}\gamma}) \mathrm{e}^{2\nu^{\mathrm{s}} L}$$

for $\gamma \in [0, 2\pi]$, upon omitting terms of higher order. In particular, the wave trains are spectrally stable for all large $L$ if $\langle v_0, w_0 \rangle M < 0$ and spectrally unstable for all large $L$ if $\langle v_0, w_0 \rangle M > 0$.

The second generic case is that the leading eigenvalues of $\partial_u f(0, c_0)$ are complex conjugate and simple. We again assume that their real part is negative (see Figure 10b). Hence, a pair of simple complex-conjugate eigenvalues $\nu^{\mathrm{s}}, \overline{\nu^{\mathrm{s}}} \in \mathrm{spec}(\partial_u f(0, c_0))$ exists such that $|\operatorname{Re} \nu| > -\operatorname{Re} \nu^{\mathrm{s}} > 0$ for every $\nu \in \mathrm{spec}(\partial_u f(0, c_0))$ with $\nu \neq \nu^{\mathrm{s}}, \overline{\nu^{\mathrm{s}}}$. We assume that $\operatorname{Im} \nu^{\mathrm{s}} \neq 0$. Exploiting the expansion

$$q'(\xi) = \operatorname{Re}[\mathrm{e}^{\nu^{\mathrm{s}} \xi} v_0] + \mathrm{O}(\mathrm{e}^{-(|\operatorname{Re} \nu^{\mathrm{s}}| + \delta)\xi}), \qquad \psi(-\xi) = \operatorname{Re}[\mathrm{e}^{\nu^{\mathrm{s}} \xi} w_0] + \mathrm{O}(\mathrm{e}^{-(|\operatorname{Re} \nu^{\mathrm{s}}| + \delta)\xi}) \tag{5.13}$$

for $\xi \to \infty$ (see [34, Chapter 3.8] or [83, Chapter X.13]), we end up with the expression

$$\lambda = \frac{a}{M} \sin(2L \operatorname{Im} \nu^{\mathrm{s}} + b)(1 - \mathrm{e}^{-\mathrm{i}\gamma}) \mathrm{e}^{2L \operatorname{Re} \nu^{\mathrm{s}}}$$

for the circle of critical eigenvalues near $\lambda = 0$. Here, $a$ and $b$ are certain real constants. Hence, if $a \neq 0$, which is equivalent to $v_0 \neq 0$ and $w_0 \neq 0$, then the periodic wave trains change their PDE stability periodically in $L$ regardless of the signs of $a$ and $M$. This is not too surprising as it is known that, in the case shown in Figure 10b, the periodic orbits to (5.1) undergo many saddle-node and period-doubling bifurcations as $L$ is varied [170, 41]. At each such bifurcation, the circle of critical eigenvalues crosses through the imaginary axis, and the wave trains either stabilize or destabilize: in the case of a period-doubling, for instance, (5.5) exhibits solutions at $\lambda = 0$ for both $\gamma = 0$ and $\gamma = \pi$. Whereas the solution with $\gamma = 0$ is enforced by the translation symmetry, the eigenvalue with $\gamma = \pi$ should cross the imaginary axis upon unfolding the period-doubling bifurcation.

Analogous results are true for wave trains near symmetric pulses provided (5.1) is reversible (see [156] for details and applications).

The above results have been applied in [156] to the wave trains that accompany the fast pulse in the FitzHugh–Nagumo equation as well as to wave trains that arise in the fourth-order equation from Example 2.

## 5.2 Multi-bump pulses

In this section, we discuss the stability of $\ell$-pulses that consist of $\ell$ well separated copies of the primary pulse. Recall that, if the primary pulse is spectrally stable with only one simple eigenvalue at $\lambda = 0$ and with the rest of the spectrum in the open left half-plane, then the spectrum of an $\ell$-pulse contains $\ell$ critical eigenvalues near $\lambda = 0$, and the rest of the spectrum is again contained in the open left half-plane, bounded away from the imaginary axis. The next theorem provides a method of computing the location of the $\ell$ critical eigenvalues.

**Theorem 5.3** ([146]) *Assume that Hypotheses 5.1 and 5.2 are met, then a $\delta > 0$ with the following properties exists. Assume that $q_\ell(\xi)$ is an $\ell$-pulse with wave speed $c_\ell$ and distances $2L_1, \dots, 2L_{\ell-1}$ such that (5.4) is met. The equation*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = [\partial_u f(q_\ell(\xi), c_\ell) + \lambda B(\xi)]u$$

*has a bounded non-zero solution $u(\xi)$ for $|\lambda| < \delta$ if, and only if,*

$$D(\lambda) = \det[A - M\lambda + R(\lambda)] = 0.$$

*The $\ell \times \ell$ matrix $A$ is tridiagonal and given by*

$$A = \begin{pmatrix} -a_1 & a_1 & & & \\ -b_1 & b_1 - a_2 & a_2 & & \\ & -b_2 & b_2 - a_3 & a_3 & \\ & & \ddots & \ddots & \\ & & & -b_{\ell-1} & b_{\ell-1} \end{pmatrix}$$

*with*

$$a_j = \langle \psi(L_j), q'(-L_j) \rangle, \qquad b_j = \langle \psi(-L_j), q'(L_j) \rangle$$

*for $j = 1, \dots, \ell - 1$. The remainder term $R(\lambda)$ is analytic in $\lambda$ and satisfies*

$$R(\lambda) = \mathrm{O}(|\lambda|\mathrm{e}^{-\rho L} + \mathrm{e}^{-3\rho L})$$

*where $L = \min_{j=1,\dots,\ell-1}\{L_j\}$.*

We remark that [146] contains a stronger result with better estimates for the remainder term $R(\lambda)$. The proof of Theorem 5.3 can be found in [146] (see also [147] for a less technical proof in $\mathbb{R}^2$). The idea of the proof is as outlined in Section 5.1.1 except that we get matching conditions for each individual pulse. The theorem can be generalized to the situation where the eigenvalue $\lambda = 0$ of the primary pulse has higher geometric multiplicity due to additional continuous symmetries (see [148]).

Theorem 5.3 states that the critical eigenvalues of the PDE linearization about an $\ell$-pulse are given as the eigenvalues of the tridiagonal matrix $A$, up to the factor $M^{-1}$, provided we ignore the remainder term $R(\lambda)$. The matrix $A$ can be thought of as the restriction of the PDE operator to the generalized eigenspace belonging to the critical eigenvalues. Note that

$$Av = 0, \qquad v = (1, \dots, 1)^* \tag{5.14}$$

so that the vector $(1, \dots, 1)^*$ corresponds to the translation eigenvalue $\lambda = 0$ of the $\ell$-pulse. The entries of the matrix $A$ are again scalar products similar to those encountered for the wave trains in Section 5.1. The only information needed to compute the entries of $A$ are the distances $2L_j$ between consecutive pulses in the $\ell$-pulse.

### 5.2.1 Strategies for using Theorem 5.3

Hence, we first comment on a general approach, namely Lin's method [111, 143], towards existence of multi-bump pulses that provides us in particular with the distances of bifurcating multi-bump pulses. Afterwards, we illustrate how the eigenvalues of $A$ can be computed using the information gathered from the existence results.

It has been proved in [111, 143] that an $\ell$-pulse with wave speed $c$ and distances $\{2L_j\}_{j=1,\dots,\ell-1}$ bifurcates from the primary pulse if, and only if, the equation

$$M(c-c_0) = \langle \psi(-L_{j-1}), q(L_{j-1})\rangle - \langle \psi(L_j), q(-L_j)\rangle + \mathrm{O}(\mathrm{e}^{-3\rho L}) \tag{5.15}$$

is satisfied for $j = 1, \dots, \ell$, where we set $L_0 = L_\ell = \infty$ and $L = \min\{L_j\}$. We remark that [143, 150] contain more general results with better estimates for the remainder term. We refer to Section 5.1.1 for the idea that leads to (5.15).

Thus, once the tails of the pulse $q(\xi)$ and the adjoint solution $\psi(\xi)$ are known, equation (5.15) can be used to investigate the existence of multi-bump pulses. If such pulses exist, their half-distances $L_j$ can be fed into the matrix

$$A = \begin{pmatrix} -a_1 & a_1 & & & \\ -b_1 & b_1-a_2 & a_2 & & \\ & -b_2 & b_2-a_3 & a_3 & \\ & & \ddots & \ddots & \\ & & & -b_{\ell-1} & b_{\ell-1} \end{pmatrix} \tag{5.16}$$

with

$$a_j = \langle \psi(L_j), q'(-L_j)\rangle, \qquad b_j = \langle \psi(-L_j), q'(L_j)\rangle$$

that determines the critical PDE eigenvalues and thus stability of the multi-bump pulses.

Before we explore examples, we comment on general strategies for solving (5.15) and for computing the eigenvalues of the tridiagonal matrix $A$ in (5.16). Let $\nu^{\mathrm{s}} < 0$ and $\nu^{\mathrm{u}} > 0$ denote the real parts of the leading stable and unstable eigenvalues of the matrix $\partial_u f(0, c_0)$. Assuming that the leading eigenvalues are semi-simple, we have

$$\langle \psi(L), q(-L)\rangle = \mathrm{O}(\mathrm{e}^{-2\nu^{\mathrm{u}} L}), \qquad \langle \psi(-L), q'(L)\rangle = \mathrm{O}(\mathrm{e}^{2\nu^{\mathrm{s}} L}) \tag{5.17}$$

(see Section 5.1.2). Typically, we have either $|\nu^{\mathrm{s}}| > \nu^{\mathrm{u}}$ or $|\nu^{\mathrm{s}}| < \nu^{\mathrm{u}}$ so that one of the two scalar products in (5.17) is of higher order. Using appropriate scalings and utilizing the implicit-function theorem, it can be verified that the higher-order scalar product can be dropped from the existence equation (5.15) and from the matrix $A$ in (5.16) (see [145, 146, 150]). The remaining equation is then much easier to analyse. Note that dropping either the entries $a_j$ for $|\nu^{\mathrm{s}}| < \nu^{\mathrm{u}}$ or else the entries $b_j$ for $|\nu^{\mathrm{s}}| > \nu^{\mathrm{u}}$ makes the matrix $A$ either superdiagonal or subdiagonal: in either case, its eigenvalues are given by the entries on the diagonal, and stability of the multi-bump pulses can be determined by inspecting the signs of the scalar products $b_j$ or $a_j$ (see again [145, 146, 150]).

We emphasize that the stability matrix $A$ is sometimes truly tridiagonal: for instance, if the underlying PDE features the reflection symmetry $x \mapsto -x$, so that the travelling-wave ODE is reversible, and the pulse is symmetric, then the stability matrix $A$ is symmetric with $a_j = -b_j$ [146]. Owing to the property (5.14), the signs of the eigenvalues of $A$ can still be determined from the signs of the elements $a_j = -b_j$ (see [146, Section 5]). Note that reversibility implies in particular that $|\nu^{\mathrm{s}}| = \nu^{\mathrm{u}}$.

We illustrate this approach by an example, namely multi-bump pulses for primary pulses $q(\xi)$ that approach saddle-focus or bifocus equilibria as $|\xi| \to \infty$. Hence, we assume that the leading spatial eigenvalues of the matrix $\partial_u f(0, c_0)$ are a pair of simple, complex-conjugate eigenvalues $\nu_*, \overline{\nu_*}$. Under this condition, infinitely many $\ell$-pulses bifurcate for each fixed $\ell > 1$. More precisely, there is a number $L_* \gg 1$ with the following property. For each choice of integers $k_1, \dots, k_{\ell-1} \in \mathbb{N}$, there is an integer $k_* \in \mathbb{N}$ such that a unique $\ell$-pulse with half-distances $L_j = L_* + (2k + k_j)\pi/|\operatorname{Im}\nu_*|$ exists, for a unique wave speed close to $c_0$, for every integer

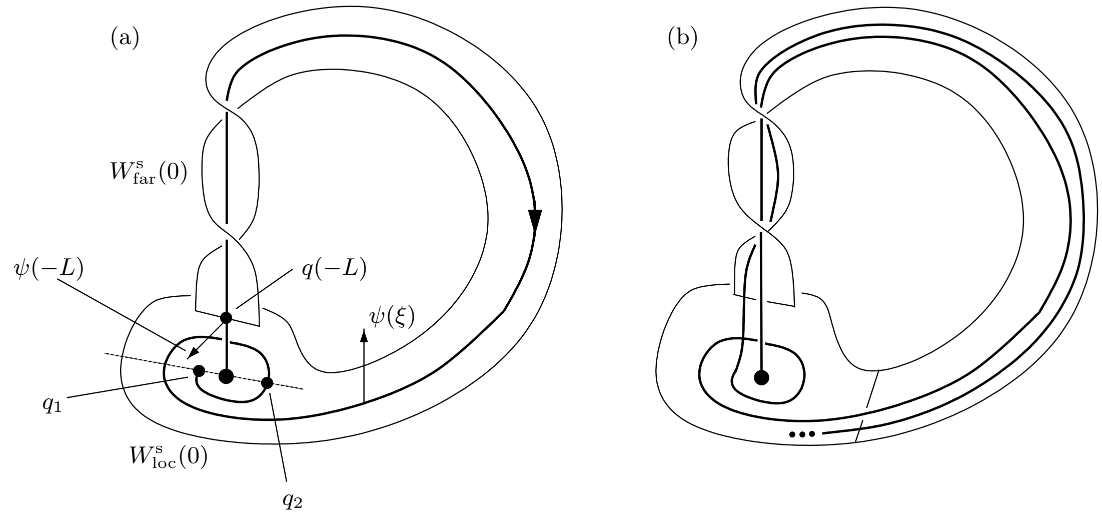


**Figure 11** This plot illustrates the existence equation (5.15) and the stability result for 2-pulses that bifurcate from a primary pulse to a saddle-focus equilibrium in $\mathbb{R}^3$ (see the main text).

$k > k_*$. The stability properties of the $\ell$-pulse described above are determined by the integers $k_j$ chosen above: define

$$k_- = \#\{j;\ k_j \text{ is even}\}, \qquad k_+ = \#\{j;\ k_j \text{ is odd}\},$$

so that $k_- + k_+ + 1 = \ell$, then the $\ell$-pulse has a simple critical eigenvalue at $\lambda = 0$, $k_-$ stable and $k_+$ unstable critical eigenvalues, or vice versa (whether $k_+$ equals the number of stable or unstable eigenvalues depends on the definition of $L_*$; see [146] for more details). These statements can be proved upon substituting the expansions (5.13) into (5.15) to get

$$M(c - c_0) = a \sin(2L_j |\operatorname{Im} \nu_*| + b)\mathrm{e}^{-2L_j|\operatorname{Re}\nu_*|}$$

(with either $j = 0, \ldots, \ell - 1$ or $j = 1, \ldots, \ell$), upon neglecting terms of higher order. The eigenvalues of the stability matrix $A$ can then be computed as outlined above, and we refer to [146, Section 6] for the details of the proof. Instead of giving these details, we focus on a fictitious travelling-wave ODE in $\mathbb{R}^3$ and explore the geometric meaning of the existence equation (5.15) and the stability results mentioned above for 2-pulses (see Figure 11). Thus, suppose that we have two stable spatial eigenvalues $\nu^{\mathrm{s}} \neq \overline{\nu^{\mathrm{s}}}$ and one unstable spatial eigenvalue $\nu^{\mathrm{u}}$ with $0 < -\operatorname{Re}\nu^{\mathrm{s}} < \nu^{\mathrm{u}}$. To see why 2-pulses exist for nearby wave speeds, we consider Figure 11a. First, we follow the two-dimensional local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(0)$ backward in $\xi$ to get the manifold $W^{\mathrm{s}}_{\mathrm{far}}(0)$, which is again close to the equilibrium at $u = 0$. Suppose we seek a 2-pulse with distance $2L$ between the two pulses: We shall follow the primary pulse, starting at $\xi = -\infty$, until we reach $q(L)$. We then vary the wave speed slightly such that the solution takes off and leaves $W^{\mathrm{s}}_{\mathrm{loc}}(0)$ to get caught by the manifold $W^{\mathrm{s}}_{\mathrm{far}}(0)$ near $q(-L)$ (see Figure 11b). To examine for which values of $L$ this approach works, we observe that the distance between $W^{\mathrm{s}}_{\mathrm{loc}}(0)$ and the lower boundary of $W^{\mathrm{s}}_{\mathrm{far}}(0)$ at $q(-L)$ is much smaller than the distance between $q(L)$ and $u = 0$ because of our assumption that $|\operatorname{Re}\nu^{\mathrm{s}}| < \nu^{\mathrm{u}}$. Thus, one should think of the lower boundary of $W^{\mathrm{s}}_{\mathrm{far}}(0)$ at $q(-L)$ as being extremely close to $W^{\mathrm{s}}_{\mathrm{loc}}(0)$ (this is not shown well in Figure 11a). For fixed $L$, we can catch the solution that leaves $W^{\mathrm{s}}_{\mathrm{loc}}(0)$ at $q(L)$ using $W^{\mathrm{s}}_{\mathrm{far}}(0)$ provided $q(L)$ is directly underneath $W^{\mathrm{s}}_{\mathrm{far}}(0)$, i.e. provided $q(L)$ lies on the dotted line in Figure 11a. This means precisely that $\langle \psi(-L), q(L) \rangle = 0$ since $\psi(-L)$ is perpendicular to $W^{\mathrm{s}}_{\mathrm{far}}(0)$ at $q(-L)$. Thus, if we have $q(L) = q_1$ or $q(L) = q_2$ (or if $q(L)$ is any other intersection point of the pulse with the dotted line), then we expect that there is a 2-pulse with distance $2L$ for a slightly perturbed wave speed (see Figure 11b). Next, we discuss the sign of the critical non-zero eigenvalue of the 2-pulse. By Theorem 5.3, the non-zero eigenvalue of the stability matrix $A$ is given by $b_1 = \langle \psi(-L), q'(L) \rangle$. Hence, if $q(L) = q_1$, then $b_1 < 0$, whereas if $q(L) = q_2$, then $b_1 > 0$. This discussion provides some insight as to the geometric meaning of the entries of the stability matrix $A$.

We refer to Section 5.2.4 for references where the approach outlined here has been used to analyse existence and stability of multi-bump pulses.

### 5.2.2 An alternative approach using the Evans function

A different approach to determining the critical PDE eigenvalues of multi-bump pulses is to compute the Evans function $D_\ell(\lambda)$ of an $\ell$-pulse and to calculate its $\ell$ zeros near $\lambda = 0$.

For 2-pulses, the idea is to calculate the derivative $D_2'(0)$: since there is only one non-zero root of $D_2(\lambda)$ near $\lambda = 0$, we can determine the sign of this root from the sign of the derivative $D_2'(0)$ utilizing a parity argument as in Section 4.2. We refer to [4, 5, 122, 123] for further details.

Nii [124] used topological indices on projective spaces to compute all roots of the Evans function $D_\ell(\lambda)$ for $\ell$-pulses that bifurcate from doubly-twisted heteroclinic loops; these multi-bump pulses exist for any $\ell > 1$. (This stability result has also been obtained in [147] using Theorem 5.1). The approach using indices appears to be restricted to problems where the critical PDE eigenvalues are real. This makes a restriction to real $\lambda$ possible and allows the use of indices.

We remark that the function $D(\lambda)$ that appears in Theorem 5.1 is the Evans function $D_\ell(\lambda)$ of the $\ell$-pulse. In many cases, it is more convenient to compute the critical eigenvalues of the stability matrix $A$ directly rather than computing the roots of the determinant $\det[A - M\lambda]$: the matrix $A$ often exhibits a special structure (such as being superdiagonal or symmetric), which simplifies the computation of its eigenvalues, but this structure may not be visible in the determinant.

### 5.2.3 Fronts and backs

Up to now, we had looked into the stability of multi-bump pulses that bifurcate from a primary pulse. Frequently, multi-bump pulses can also be constructed by gluing fronts and backs together. In other words, they may bifurcate from heteroclinic loops that consist of connections between two different equilibria $p_1$ and $p_2$ (see Figure 12). We need to distinguish between two different scenarios.

If the equilibria $p_1$ and $p_2$ are such that

$$\dim W^{\mathrm{u}}(p_1) = \dim W^{\mathrm{u}}(p_2),$$

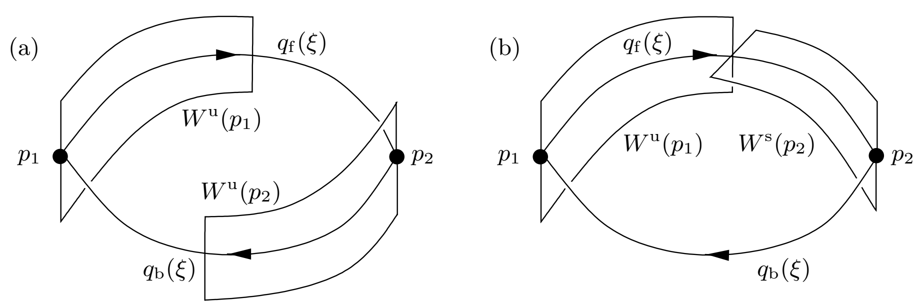


**Figure 12** Two heteroclinic loops comprised of a front $q_{\mathrm{f}}(\xi)$ and a back $q_{\mathrm{b}}(\xi)$. The plotted manifolds are the stable and unstable manifolds of the respective equilibria.

which is the geometric configuration shown in Figure 12a, then Theorem 5.1 and the results reviewed in Sections 5.2.1 and 5.2.2 readily generalize to cover multi-bump pulses that consist of several alternating copies of the front and the back. We refer to [123, 124, 147] for theory and applications.

If, on the other hand, the geometry is as shown in Figure 12b, so that

$$\dim W^{\mathrm{u}}(p_1) = \dim W^{\mathrm{u}}(p_2) + 1 \qquad \text{or} \qquad \dim W^{\mathrm{u}}(p_1) = \dim W^{\mathrm{u}}(p_2) - 1,$$

then the situation is more complicated: since the Morse indices of the equilibria $p_1$ and $p_2$ differ, one of them, say $p_2$, has essential spectrum in the right half-plane. Thus, on account of the results in Section 3.4, both the front and the back are unstable. The theory outlined above is then no longer applicable. Nevertheless, it has recently been proved that the multi-bump pulses that converge to the stable equilibrium $p_1$ can be stable even though front and back are both unstable (see [125, 154]). The reason for this unexpected behaviour is that the essential spectrum behaves rather strangely under matching or gluing [155]. We also refer to Section 6.3.2 for a related phenomenon.

### 5.2.4 A review of existence and stability results of multi-bump pulses and applications

Over the past decades, bifurcations to multi-bump pulses, and wave trains, have been the subject of numerous articles. Summaries of relevant results can be found in [28, 55]. Here, we focus on those bifurcations for which the stability of the bifurcating multi-bump pulses has been analyzed.

In Section 5.2.1, we have already mentioned Shilnikov's saddle-focus and bifocus bifurcation. The existence of multi-bump pulses has been studied in [170, 73] (see also [41, 111]). The stability of the bifurcating 2-pulses, and certain 3-pulses, has been investigated in [4, 5, 187]. Stability results for $\ell$-pulses with arbitrary $\ell > 1$ can be found in [146].

Next, suppose that the travelling-wave ODE is reversible, which is the case when the underlying PDE exhibits the reflection symmetry $x \mapsto -x$, and that the primary pulse $q(x)$ is reflection invariant. If the asymptotic equilibrium $u = 0$ towards which the pulse converges is a bifocus, i.e. if it has non-real simple leading eigenvalues, then again infinitely many $\ell$-pulses exist for each $\ell > 1$ [25, 27, 42, 82]. The stability of these multi-bump pulses has been analyzed in [145]. We refer to the theme issue [174] for applications.

There are a number of bifurcations that require two parameters (the wave speed $c$ and an additional system parameter) to be encountered and properly unfolded. Among them are the resonant, the inclination-flip and the orbit-flip bifurcation. All of these bifurcations lead, under appropriate conditions, to multi-bump pulses [186, 32, 86, 104, 143]. The stability of these pulses is studied in [126] utilizing Theorem 5.3.

As mentioned in Section 5.2.3, doubly-twisted heteroclinic loops also lead to multi-bump pulses [39]: their stability has been investigated in [124, 147]. This bifurcation occurs in the FitzHugh–Nagumo equation [40], and the resulting multi-bump pulses where found to be stable in [124, 147].

Lastly, we again consider reversible travelling-wave ODEs. Bifurcations of codimension one that lead to multi-bump pulses include the reversible orbit-flip [150] and the semi-simple bifurcation [188]. The stability of $\ell$-pulses that bifurcate at reversible orbit-flips has been analyzed in [150], and applications to fourth-order equations and parametrically-forced NLS equations that model optical fibers under phase-sensitive amplification can be found in [150] and [97], respectively. The instability of $\ell$-pulses to coupled NLS equations that admit a semi-simple bifurcation has been investigated in [189, 190].

## 5.3 Weak interaction of pulses

One interesting feature of the stability matrix $A$ that appears in Theorem 5.3 is that this matrix is tridiagonal. Since the $j$th row of the matrix $A$ is associated with the translation eigenfunction $Q'(\cdot - \xi_j)$ of the $j$th pulse in the multi-bump pulse, located at position $\xi_j \in \mathbb{R}$, it appears as if the individual pulses interact only with their nearest neighbours. This is indeed the case: Suppose that we substitute the initial condition

$$U_0(\xi) = \sum_{j=1}^{\ell} Q(\xi - \xi_j)$$

into the PDE, where $\xi_j \in \mathbb{R}$ denotes the position of the $j$th pulse (see Figure 13). We call such a function a pulse packet provided the distances between consecutive pulses are large, i.e. provided $\xi_{j+1} - \xi_j \gg 1$ for $j = 1, \ldots, \ell - 1$. If we solve the PDE with initial condition $U_0(\xi)$, then it turns out that the shape of each individual pulse in the pulse packet is maintained; the time-dependence of the solution manifests itself only in the movement of the position of each pulse. In other words, the solution $U(\xi, t)$ is, to leading order, given by

$$U(\xi, t) = \sum_{j=1}^{\ell} Q(\xi - \xi_j(t)) \tag{5.18}$$

where the positions $\xi_j(t)$ depend upon the time variable $t$. Using the above ansatz, it is also possible to derive ODEs that govern the evolution of the positions $\xi_j(t)$ (see [52, 117, 130, 51, 149]). The ODE that describes the interaction of the pulses in the pulse packet can be written as [149]

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_j = \frac{1}{M}\Big(\langle \psi(-L_{j-1}), q(L_{j-1})\rangle - \langle \psi(L_j), q(-L_j)\rangle\Big) + \mathrm{O}(\mathrm{e}^{-3\rho L}) \tag{5.19}$$

where

$$L_0 = L_\ell = \infty, \qquad L_j = \frac{\xi_{j+1} - \xi_j}{2}, \qquad j = 1, \ldots, \ell - 1,$$

and $L = \min_{j=1,\ldots,\ell-1}\{L_j\}$. The interested reader may wish to compare this equation to the existence equation (5.15) for multi-bump pulses that was derived in [111, 143]: the term $c - c_0$ in (5.15), which is the wave speed of the $j$th pulse in an $\ell$-pulse (measured relative to the primary pulse), is replaced by the wave speed $\partial_t \xi_j(t)$ of the $j$th pulse in a wave packet (again relative to the speed of the primary pulse). Hence, once we know that the time evolution of pulse packets is, to leading order, given by (5.18), then the interaction equation (5.19) can be derived by using Lin's method as outlined in Section 5.1.1. Equation (5.18) can be confirmed by proving the existence of a center manifold for the underlying PDE that is formed by pulse packets. This requires to establish normal hyperbolicity (which is a consequence of the stability results in [146]) and to utilize a cut-off function that acts only within a finite-dimensional approximation of the center manifold. We refer to [149] for details.

Equation (5.19) has been derived rigorously in [51] using Liapunov–Schmidt reduction for the PDE and, simultaneously and independently, in [149] using a center-manifold reduction and subsequent Liapunov–Schmidt reduction for the flow on the center manifold. In fact, [149] contains an improved version of (5.19)

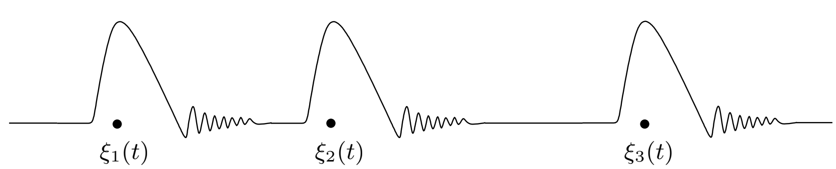


**Figure 13** A pulse packet consisting of three identical, well separated pulses. The positions of the pulses are denoted by the time-dependent coordinates $\xi_1, \xi_2, \xi_3$.

that is applicable near homoclinic bifurcations of codimension two. We also refer to [26, 62] for earlier results on the interaction of meta-stable patterns in scalar reaction-diffusion equations. Lastly, we mention that there are interesting relations between the interaction equation (5.19) and the nonlinear dispersion relation $c(L)$ that relates the wave speed $c$ and the wavelength $L$ of the wave trains that accompany the pulse (see e.g. [117, 130]).

# 6 Numerical computation of spectra

In many applications, it appears to be impossible, or at least very difficult, to investigate the existence and stability of travelling waves by analytical means. In such a situation, numerical computations are often the only way to obtain information about travelling waves. In this section, we summarize some theoretical results in this direction and provide pointers to algorithms and numerical software for the numerical computation of waves and their PDE spectra.

## 6.1 Continuation of travelling waves

Pulses, fronts, and wave trains can be continued numerically as certain system parameters are varied, once a good starting solution is available for one set of parameter values (see the survey [13]).

For pulses, the idea is to approximate the condition for having a pulse, namely that the pulse is contained in the unstable and stable manifolds of the equilibrium at $u = 0$ (see [13, 33, 80, 107]), i.e.

$$u(-L) \in W^{\mathrm{u}}_{\mathrm{loc}}(0), \qquad u(L) \in W^{\mathrm{s}}_{\mathrm{loc}}(0),$$

by a condition that is posed on a finite interval $[-L, L]$ to make it computable. For instance, if $q(\xi)$ denotes the exact pulse, then a numerical approximation can be sought as a solution to the travelling-wave ODE

$$u' = f(u, c) \tag{6.1}$$

on the interval $(-L, L)$ that satisfies the boundary conditions

$$u(-L) \in T_0 W^{\mathrm{u}}_{\mathrm{loc}}(0) = E^{\mathrm{u}}_0, \qquad u(L) \in T_0 W^{\mathrm{s}}_{\mathrm{loc}}(0) = E^{\mathrm{s}}_0 \tag{6.2}$$

together with the phase condition

$$\int_{-L}^{L} \langle q'(\xi), u(\xi) - q(\xi) \rangle \, \mathrm{d}\xi = 0, \tag{6.3}$$

which breaks the translation invariance and singles out a specific translate. Here, $E^{\mathrm{s}}_0$ and $E^{\mathrm{u}}_0$ denote the generalized stable and unstable eigenspaces of $\partial_u f(0, c_0)$. We remark that the exact pulse $q(\xi)$ that appears in the phase condition (6.3) can be replaced by any reasonable guess for $q(\xi)$. We refer to [12, 61] for algorithms related to the continuation of pulses and fronts.

Analogously, periodic waves of period $2L$ can be sought as solutions to (6.1) and (6.3) together with the boundary condition

$$u(L) = u(-L).$$

These algorithms can be implemented in boundary-value solvers such as AUTO97 (see [44]). In fact, AUTO97 computes the boundary conditions (6.2) for pulses automatically and also detects various homoclinic bifurcations that lead to multi-bump pulses (see [29, 44]). We refer to the survey [13] for more details related to the computation, and continuation, of travelling waves and to [8] as a general reference for numerical methods for boundary-value problems.

## 6.2 Computation of spectra of spatially-periodic wave trains

Suppose that $Q(\xi)$ is a wave train with period $L$ so that $Q(\xi + L) = Q(\xi)$ for all $\xi$. We had seen in Section 3.4.2 that $\lambda$ is in the spectrum $\Sigma$ of the linearization about $Q(\xi)$ if, and only if, the boundary-value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}u &= A(\xi;\lambda)u, \qquad 0 < \xi < L \\ u(L) &= \mathrm{e}^{\mathrm{i}\gamma}u(0) \end{aligned} \tag{6.4}$$

has a solution $u(\xi)$ for some $\gamma \in \mathbb{R}$. One possible numerical procedure to find all solutions to (6.4) is as follows.

First, compute all solutions to (6.4) for $\gamma = 0$. This can be done by discretizing the operator $\mathcal{L}$ (or $\mathcal{T}$) with periodic boundary conditions using, for instance, finite differences or pseudo-spectral methods [8], and to compute the spectrum of the resulting large matrix using eigenvalue-solvers (see e.g. [7]). Note that, if we restrict to $\gamma = 0$, equation (6.4) describes precisely the eigenvalues of the wave train under periodic boundary conditions $u(0) = u(L)$.

Second, once we have calculated all eigenvalues for $\gamma = 0$, we can utilize continuation codes (e.g. AUTO97 [44]) to compute the solutions to (6.4) for $\gamma \neq 0$ by using path-following of the solutions for $\gamma = 0$ in $\gamma$. We refer to [159] for an example where this procedure has been carried out successfully.

The advantage of this approach is that the spectrum is computed with high accuracy. Also, since the most interesting eigenvalues are those close to the imaginary axis or in the right half-plane, one would need to continue only a few relevant eigenvalues in $\gamma$.

We remark that the approach outlined above gives all eigenvalues only if, for each solution $(\gamma_*, \lambda_*)$ of (6.4), there is a continuous curve $(\gamma, \lambda(\gamma))$ of solutions to (6.4), parametrized by $\gamma \in [0, \gamma_*]$, such that $\lambda_* = \lambda(\gamma_*)$. If there is an island of solutions that is not connected to any eigenvalue at $\gamma = 0$, then this island could never be reached by continuation in $\gamma$. Fortunately, it is possible to prove that, for reaction-diffusion systems, such islands cannot exist: For bounded islands, this is a consequence of winding-number type arguments using the analyticity of the Evans function $D_{\mathrm{per}}(\gamma, \lambda)$ in $(\gamma, \lambda)$ (see (3.17)). Unbounded islands can be excluded upon using scaling arguments as in Section 4.2.2.

## 6.3 Computation of spectra of pulses and fronts

We consider the operators $\mathcal{T}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi;\lambda)$ and the associated eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u = A(\xi;\lambda)u \tag{6.5}$$

(see Section 3). Suppose that $Q(\xi)$ is either a front or a pulse, so that there are $n \times n$ matrices $A_\pm(\lambda)$ with

$$|A(\xi;\lambda) - A_\pm(\lambda)| \leq K\mathrm{e}^{-\rho|\xi|}$$

as $\xi \to \pm\infty$ for certain positive constants $K$ and $\rho$ that are independent of $\xi$ and $\lambda$ (see Sections 3.4.3 and 3.4.4). As before, we denote by $\Sigma = \Sigma_{\mathrm{pt}} \cup \Sigma_{\mathrm{ess}}$ the various spectra associated with $\mathcal{T}$. We are interested in computing the spectrum using periodic boundary conditions (for pulses) or separated boundary conditions (for pulses or fronts).

### 6.3.1 Periodic boundary conditions

Suppose that $A_+(\lambda) = A_-(\lambda)$ for all $\lambda$. We consider the operator

$$\mathcal{T}_L^{\rm per}(\lambda) \;:\qquad H^1_{\rm per}((-L,L),\mathbb{C}^n) \longrightarrow L^2((-L,L),\mathbb{C}^n), \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u$$

where the function space

$$H^1_{\rm per}((-L,L),\mathbb{C}^n) = H^1((-L,L),\mathbb{C}^n) \cap \{u;\ u(-L) = u(L)\},$$

encodes periodic boundary conditions. We denote by $\Sigma_L^{\rm per}$ the spectrum of $\mathcal{T}_L^{\rm per}$ and observe that $\Sigma_L^{\rm per}$ consists entirely of point spectrum (see e.g. [155]). The next theorem states that $\Sigma_L^{\rm per}$ converges to $\Sigma$ as $L \to \infty$ uniformly in bounded subsets of $\mathbb{C}$.

**Theorem 6.1** *Assume that Hypothesis 5.3 is met.*

- *Fix an eigenvalue $\lambda_*$ with multiplicity $\ell$ in $\Sigma_{\rm pt}$. As $L \to \infty$, there are precisely $\ell$ elements in $\Sigma_L^{\rm per}$, counted with multiplicity, close to $\lambda_*$, and these elements converge to $\lambda_*$ as $L \to \infty$. In other words, isolated eigenvalues of the pulse are approximated by elements in $\Sigma_L^{\rm per}$, counting multiplicity [14, 155].*
- *Fix $\lambda_* \in \Sigma_{\rm ess}$, then, under an additional technical assumption ([155, Hypothesis 6]), $\lambda_*$ is approached by infinitely many eigenvalues in $\Sigma_L^{\rm per}$ as $L \to \infty$ [155].*
- *Fix a bounded domain $\Omega \subset \mathbb{C}$. For any $\delta > 0$, an $L_*$ exists such that $(\Sigma_L^{\rm per} \cap \Omega) \subset \mathcal{U}_\delta(\Sigma)$ for all $L > L_*$ [155].*

See, for instance, [191] for numerical computations that corroborate this statement.

### 6.3.2 Separated boundary conditions

Recall that we consider a pulse or a front. Separated boundary conditions can be realized by choosing appropriate subspaces $E_+^{\rm bc}$ and $E_-^{\rm bc}$ of $\mathbb{C}^n$. We then consider the operator

$$\mathcal{T}_L^{\rm sep}(\lambda) \;:\qquad H^1_{\rm sep}((-L,L),\mathbb{C}^n) \longrightarrow L^2((-L,L),\mathbb{C}^n), \qquad u \longmapsto \frac{\mathrm{d}u}{\mathrm{d}\xi} - A(\cdot;\lambda)u,$$

where the function space

$$H^1_{\rm sep}((-L,L),\mathbb{C}^n) = H^1((-L,L),\mathbb{C}^n) \cap \{u;\ u(-L) \in E_-^{\rm bc} \text{ and } u(L) \in E_+^{\rm bc}\},$$

encodes separated boundary conditions.

**Hypothesis 6.1** *We assume that the following conditions are met.*

- *A number $\rho > 0$ and an integer $i_\infty \in \mathbb{N}$ exist such that, for all $\lambda$ with $\operatorname{Re}\lambda \geq \rho$, the asymptotic matrices $A_\pm(\lambda)$ are hyperbolic, and the dimension of their generalized unstable eigenspaces is equal to $i_\infty$.*
- *The subspaces $E_\pm^{\rm bc}$ satisfy $\dim E_-^{\rm bc} = \operatorname{codim} E_+^{\rm bc} = i_\infty$.*

In other words, for separated boundary conditions, the integer $i_\infty$ is singled out as the number of boundary conditions at the right endpoint of the interval $(-L, L)$; observe that the number of boundary conditions at $\xi = \pm L$ is equal to the codimension of $E_\pm^{\rm bc}$. The integer $i_\infty$ is also equal to the asymptotic Morse index of the matrices $A_\pm(\lambda)$ as $\operatorname{Re}\lambda \to \infty$.

We denote by $\Sigma_L^{\rm sep}$ the spectrum of $\mathcal{T}_L^{\rm sep}$ and remark that $\Sigma_L^{\rm sep}$ consists entirely of point spectrum (see e.g. [155]). It turns out that the spectrum $\Sigma_L^{\rm sep}$ does not resemble the spectrum $\Sigma$ of $\mathcal{T}$ but an, in general, entirely different set that we shall describe next.

We label eigenvalues of $A_\pm(\lambda)$ according to their real part, and repeated with their multiplicity,

$$\operatorname{Re}\nu_1^\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu_{i_\infty}^\pm(\lambda) \geq \operatorname{Re}\nu_{i_\infty+1}^\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu_n^\pm(\lambda).$$

We can now define the so-called absolute spectrum of $\mathcal{T}$ [155].

**Definition 6.1 (Absolute spectrum)** *We define* $\Sigma_{\text{abs}}^+ = \{\lambda \in \mathbb{C};\ \operatorname{Re}\nu_{i_\infty}^+(\lambda) = \operatorname{Re}\nu_{i_\infty+1}^+(\lambda)\}$ *and, analogously,* $\Sigma_{\text{abs}}^- = \{\lambda \in \mathbb{C};\ \operatorname{Re}\nu_{i_\infty}^-(\lambda) = \operatorname{Re}\nu_{i_\infty+1}^-(\lambda)\}$. *The absolute spectrum* $\Sigma_{\text{abs}}$ *of* $\mathcal{T}$ *is the union of* $\Sigma_{\text{abs}}^+$ *and* $\Sigma_{\text{abs}}^-$.

Next, suppose that $\lambda \notin \Sigma_{\text{abs}}$, so that there is a gap, in the real part, between the spatial eigenvalues of $A_\pm(\lambda)$ with indices $i_\infty$ and $i_\infty + 1$, i.e. so that

$$\operatorname{Re}\nu_1^\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu_{i_\infty}^\pm(\lambda) > \eta_\pm > \operatorname{Re}\nu_{i_\infty+1}^\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu_n^\pm(\lambda)$$

for some $\eta_\pm = \eta_\pm(\lambda)$ (see Figure 3). We denote by $\tilde{E}_\pm^{\text{u}}(\lambda)$ and $\tilde{E}_\pm^{\text{s}}(\lambda)$ the generalized eigenspaces of $A_\pm(\lambda)$ associated with the spectral sets $\{\nu_1^\pm, \ldots, \nu_{i_\infty}^\pm\}$ and $\{\nu_{i_\infty+1}^\pm, \ldots, \nu_n^\pm\}$, respectively. Owing to the presence of the spectral gap at $\operatorname{Re}\nu = \eta_\pm$, equation (6.5) has exponential dichotomies on $\mathbb{R}^\pm$ for every $\lambda \notin \Sigma_{\text{abs}}$ with projections $\tilde{P}_\pm(\xi;\lambda)$ such that $\mathrm{N}(\tilde{P}_-(\xi;\lambda)) \to \tilde{E}_-^{\text{u}}(\lambda)$ as $\xi \to -\infty$ and $\mathrm{R}(\tilde{P}_+(\xi;\lambda)) \to \tilde{E}_+^{\text{s}}(\lambda)$ as $\xi \to \infty$ (see Section 3.2). We define the analytic functions

$$D_{\text{sep}}(\lambda) := \mathrm{N}(\tilde{P}_-(0;\lambda)) \wedge \mathrm{R}(\tilde{P}_+(0;\lambda)), \qquad D_{\text{bc}}^-(\lambda) := E_-^{\text{bc}} \wedge \tilde{E}_-^{\text{s}}(\lambda), \qquad D_{\text{bc}}^+(\lambda) := E_+^{\text{bc}} \wedge \tilde{E}_+^{\text{u}}(\lambda)$$

(see Section 4.1 for this notation) that measure eigenvalues or resonance poles of the underlying wave as well as transversality of the boundary conditions with the pseudo-stable and pseudo-unstable eigenspaces that we introduced above.

**Definition 6.2 (Pseudo-point spectrum)** *We define*

$$\tilde{\Sigma}_{\text{pt}} = \{\lambda \notin \Sigma_{\text{abs}};\ \ell_{\text{sep}} + \ell_- + \ell_+ > 0\}$$

*where* $\ell_{\text{sep}}$ *and* $\ell_\pm$ *denote the order of* $\lambda$ *as a zero of the functions* $D_{\text{sep}}$ *and* $D_\pm$ *defined above. We call* $\ell = \ell_{\text{sep}} + \ell_- + \ell_+$ *the multiplicity of* $\lambda$ *for* $\lambda \in \tilde{\Sigma}_{\text{pt}}$.

The next theorem states that the spectrum $\Sigma_L^{\text{sep}}$ does not approximate the spectrum $\Sigma = \Sigma_{\text{pt}} \cup \Sigma_{\text{ess}}$ of $\mathcal{T}$ but the set $\tilde{\Sigma}_{\text{pt}} \cup \Sigma_{\text{abs}}$.

**Theorem 6.2 ([155])** *Assume that Hypotheses 5.3 and 6.1 are met.*

- *Fix an eigenvalue* $\lambda_*$ *with multiplicity* $\ell$ *in* $\tilde{\Sigma}_{\text{pt}}$. *As* $L \to \infty$, *there are precisely* $\ell$ *elements in* $\Sigma_L^{\text{sep}}$, *counted with multiplicity, close to* $\lambda_*$, *and these elements converge to* $\lambda_*$ *as* $L \to \infty$.
- *Fix* $\lambda_* \in \Sigma_{\text{abs}}$, *then, under additional technical assumptions ([155, Hypotheses 7 and 8]),* $\lambda_*$ *is approached by infinitely many eigenvalues in* $\Sigma_L^{\text{sep}}$ *as* $L \to \infty$.
- *Fix a bounded domain* $\Omega \subset \mathbb{C}$. *For any* $\delta > 0$, *there is an* $L_*$ *such that* $(\Sigma_L^{\text{sep}} \cap \Omega) \subset \mathcal{U}_\delta(\tilde{\Sigma}_{\text{pt}} \cup \Sigma_{\text{abs}})$ *for all* $L > L_*$.

Hence, eigenvalues on large bounded intervals under separated boundary conditions are created via two different mechanisms: First, eigenvalues are created whenever the spaces that encode the boundary conditions are not transverse to the pseudo-stable or pseudo-unstable eigenspaces which are related to the two spatial spectral sets associated with the number of boundary conditions. Second, eigenvalues arise as zeros of the Evans function $D_{\text{sep}}(\lambda)$ that is again related to the aforementioned pseudo-stable or pseudo-unstable eigenspaces.

We emphasize that the sets $\tilde{\Sigma}_{\text{pt}}$ and $\Sigma_{\text{pt}}$ coincide to the right of the essential spectrum $\Sigma_{\text{ess}}$ because of Hypothesis 6.1. The absolute spectrum is typically to the left of the essential spectrum. We remark that, if the underlying PDE is reflection invariant, then $\tilde{\Sigma}_{\text{pt}}$ and $\Sigma_{\text{abs}}$ are typically equal to $\Sigma_{\text{pt}}$ and $\Sigma_{\text{ess}}$, respectively, except possibly for additional eigenvalues that are created on the bounded interval through non-transverse boundary conditions.

# 7 Nonlinear stability

In this section, we consider nonlinear stability of travelling waves. Suppose that $Q(\cdot)$ denotes a travelling wave that is spectrally stable, so that the spectrum of the linearization $\mathcal{L}$ of the PDE about the wave $Q(\cdot)$ is contained in the left half-plane. We are then interested in the stability of the wave $Q(\cdot)$ for the full PDE. Since there is an entire family of waves, namely $Q(\cdot)$ together with its translates $Q(\cdot+\tau)$, we say that the wave is nonlinearly stable if, for any initial condition $U_0(\cdot)$ sufficiently close to $Q(\cdot)$, the associated solution $U(\cdot,t)$ stays near the family $\{Q(\cdot+\tau);\ \tau\in\mathbb{R}\}$ for all $t>0$. More precisely, we have the following definition.

**Definition 7.1 (Nonlinear stability)** *We say that a travelling wave $Q$ is nonlinearly stable if, for every $\epsilon>0$, there is a $\delta>0$ with the following property: if $U_0$ is an initial condition in $\mathcal{U}_\delta(Q)$, then the associated solution $U(\cdot,t)$ satisfies $U(\cdot,t)\in\mathcal{U}_\epsilon(\{Q(\cdot+\tau);\ \tau\in\mathbb{R}\})$ for $t>0$. We say that $Q$ is nonlinearly stable with asymptotic phase if, for each $U_0$ as above, a $\tau_*$ exists such that $U(\cdot,t)\to Q(\cdot+\tau_*)$ as $t\to\infty$.*

Note that we have not yet mentioned any function spaces or norms. Often, one would have to measure the various neighbourhoods that appear in Definition 7.1 in different, not necessarily equivalent, norms.

Nonlinear stability properties depend strongly on the nature of the PDE, in particular, on the properties of the linearization $\mathcal{L}$ about the travelling wave $Q$ (see Figure 14). Throughout the remainder of this section, we consider a PDE of the form

$$U_t=\mathcal{A}U+\mathcal{N}(U). \tag{7.1}$$

We assume that $\mathcal{A}:\mathcal{X}\to\mathcal{X}$ is a densely defined, closed operator, where $\mathcal{X}$ is an appropriate Banach space.

### Sectorial operators

Suppose that the operator $\mathcal{A}$ is sectorial: its resolvent set contains the sector $\{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda>a-b|\operatorname{Im}\lambda|\}$ for some $a\in\mathbb{R}$ and some $b>0$, and the resolvent of $\mathcal{A}$ satisfies an estimate of the form

$$\|(\mathcal{A}-\lambda)^{-1}\|\le\frac{K}{|\lambda-a|}$$

for $\lambda$ in the above sector. We refer to [85] for more background on sectorial operators as well as for the results mentioned below. Associated with the sectorial operator $\mathcal{A}$ are its fractional power spaces $\mathcal{X}^\alpha$: we have $\mathcal{X}^0=\mathcal{X}$ and $\mathcal{X}^1=\mathcal{D}(\mathcal{A})$, and $\mathcal{X}^\alpha$ with $\alpha\in(0,1)$ interpolates between these two spaces. If, for some $\alpha\in[0,1)$, the nonlinearity $\mathcal{N}:\mathcal{X}^\alpha\to\mathcal{X}$ is differentiable, then we can solve (7.1) in $\mathcal{X}^\alpha$ for any initial condition in $\mathcal{X}^\alpha$ (see [85, Section 1]).

Suppose that $Q(\cdot)$, together with its translates $Q(\cdot+\tau)$, is a travelling-wave solution to a PDE that can be cast in the above fashion[11]. We then have the following result [85, Section 5.1] that can be briefly stated as

[11] At this point, there is always some restriction by going from the abstract framework (7.1) to concrete applications to travelling waves where $\mathcal{A}$ is differential operator posed on a function space $\mathcal{X}$ such as $L^2(\mathbb{R},\mathbb{R}^N)$

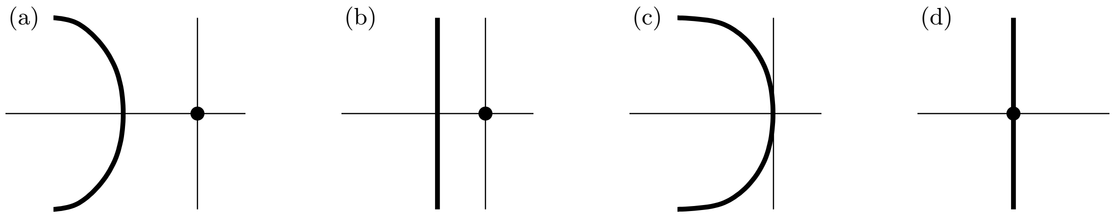


**Figure 14** The spectrum of the linearization $\mathcal{L}$ about a spectrally stable wave is shown in (a) if $\mathcal{L}$ is sectorial and in (b) if $\mathcal{L}$ generates a $C^0$-semigroup. In (c), the spectrum of the sectorial linearization about a marginally stable wave is plotted, whereas (d) contains the spectrum if $\mathcal{L} = \mathcal{J}\mathcal{H}''(Q)$ comes from a Hamiltonian PDE (see Section 8).

*spectral stability implies nonlinear stability with asymptotic phase.* More precisely, if the spectrum $\Sigma$ of the operator $\mathcal{L} = \mathcal{A} + \partial_U \mathcal{N}(Q)$, posed on $\mathcal{X}$, satisfies

$$\Sigma \setminus \{0\} \subset \{\lambda;\, \mathrm{Re}\,\lambda < -\delta\}$$

for some $\delta > 0$, and if $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}$, then the travelling wave $Q$ is nonlinearly stable with asymptotic phase (see Definition 7.1 applied to $\mathcal{X}^\alpha$). We refer to [85, Section 5.1] for the proof that uses a center-manifold reduction (see also [56, 160]).

The above result is, for instance, applicable to the reaction-diffusion system (2.10) provided the diffusion matrix is strictly positive [85].

### $\mathbf{C^0}$-Semigroups

Next, we consider the situation where the operator $\mathcal{A}$ generates only a $C^0$-semigroup on $\mathcal{X}$ (see e.g. [133, Section 1] for sufficient and necessary conditions on $\mathcal{A}$). Consequently, the PDE (7.1) has mild solutions in $\mathcal{X}$ for each initial condition in $\mathcal{X}$ provided $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is differentiable [133, Section 6].

Suppose again that $Q(\cdot)$, together with its translates $Q(\cdot + \tau)$, is a travelling-wave solution to the PDE (7.1). Assume that the linear semigroup $\mathrm{e}^{\mathcal{L}t}$ that is associated with the linearization $\mathcal{L} = \mathcal{A} + \partial_U \mathcal{N}(Q)$ and posed on $\mathcal{X}$ has a simple eigenvalue $\lambda = 1$, and the rest of its spectrum is contained inside the circle of radius $\mathrm{e}^{-\delta t}$ for $t > 0$ for some fixed $\delta > 0$. Under this assumption, the travelling wave $Q$ is nonlinearly stable with asymptotic phase. We refer to [9] for a proof that uses center-manifold reduction. This nonlinear-stability result is applicable to the reaction-diffusion system (2.10) if the diffusion matrix $D$ is non-negative; we refer to [54] and [9] for this and other applications.

The main difficulty in applying the above nonlinear-stability result is that the spectrum of the linear semigroup $\mathrm{e}^{\mathcal{L}t}$ is not necessarily computable using the spectrum of its generator $\mathcal{L}$: the spectral theorem is not true for generators of $C^0$-semigroups (see e.g. [133, Section 2.2] for counterexamples).

If, however, the generator $\mathcal{L}$ of a $C^0$-semigroup satisfies a resolvent estimate of the form

$$\|(\mathcal{L} - \lambda)^{-1}\| \le K$$

for all $\lambda$ with $\mathrm{Re}\,\lambda \ge \eta$ for some fixed $\eta \in \mathbb{R}$ and $K$, then the semigroup satisfies $\|\mathrm{e}^{\mathcal{L}t}\| \le C\mathrm{e}^{\eta t}$ owing to a result by Prüss [138, Corollary 4]. This criterion can be used to establish estimates for a semigroup using the spectral information for its generator. We refer to [97] for an application to a perturbed NLS equation; in fact, the result in [97] applies more generally to coupled NLS equation with arbitrary bounded potentials in several space dimensions.

### Essential spectrum up to $i\mathbb{R}$

We comment on various problems where the essential spectrum of the relevant operator $\mathcal{L}$ touches the imaginary axis at $\lambda = 0$ in a quadratic tangency. This situation occurs naturally when considering spatially-periodic travelling waves (see Section 3.4), shock waves in conservation laws (see [192] and references therein) or fronts that connect stable to unstable rest states (see [162]). In all these cases, it becomes necessary to introduce polynomial or even exponential weights in the space or time variables.

The nonlinear stability of spatially-periodic waves has been investigated for the Ginzburg–Landau equation [17, 35], the Swift–Hohenberg equation [166] and reaction-diffusion systems [168]. In higher space dimensions, nonlinear stability has been demonstrated for Taylor vortices [167] and for roll solutions to the Swift–Hohenberg equation [178]. We also refer to [50] where the nonlinear stability of periodic patterns in the Swift–Hohenberg equation is studied using invariant manifolds that discriminate between different algebraic temporal decay rates.

Fronts that connect stable to unstable rest states occur in many equations. Consider, for instance, the scalar reaction-diffusion equation

$$U_t = U_{xx} + F(U), \qquad x \in \mathbb{R}.$$

The aforementioned fronts arise in the Kolmogorov–Petrovsky–Piskunov (KPP) equation, where $F(U) = U(1-U)$, and in the real Ginzburg–Landau (GL) equation, where $F(U) = U(1-|U|^2)$. In fact, there is typically a continuum of fronts parametrized by their wave speeds; the wave speed of the slowest wave is denoted by $c_*$. In the non-critical case ($c \neq c_*$), it is possible to use exponential weights in the spatial direction to stabilize these fronts as demonstrated by Sattinger [162]. More refined estimates have been obtained in [93] using polynomial weights and resolvent estimates. In the critical case ($c = c_*$), nonlinear stability has been proved in [18] for the GL equation using renormalization-group techniques, in [102] for the KPP equation, and in [49], using Liapunov functionals, for general nonlinearities. In [63], optimal temporal decay estimates were obtained for general nonlinearities using renormalization-group methods [19]. We refer to [103, 139, 184] for additional references. An interesting problem in this context is which of the fronts (or, alternatively, which wave speed) is selected by a given initial condition. We refer to [48, 184] for references regarding this issue.

The nonlinear stability of fronts that connect spatially-periodic states has been investigated in [18, 64] for the real Ginzburg–Landau equation using energy estimates. Energy estimates have also been used in [65, 66] to establish the stability of fronts in damped hyperbolic equations.

Lastly, the nonlinear stability of certain viscous shocks has been demonstrated in [92, 93] using polynomial weights and resolvent estimates. More general results can be found in [192] where pointwise estimates were utilized; [192] also contains many references to different approaches towards the stability of viscous shock profiles.

# 8 Equations with additional structure

In this section, we give pointers to the literature for methods that are applicable to PDEs with additional structure such as Hamiltonian, monotone, and singularly-perturbed equations.

### Hamiltonian PDEs

Consider an abstract evolution equation of the form

$$V_t = \mathcal{J}\mathcal{E}'(V), \tag{8.1}$$

posed on a Hilbert space $\mathcal{X}$, where the differentiable functional $\mathcal{E} : \mathcal{X} \to \mathbb{R}$ is thought of as the energy, and $\mathcal{J}$ is a skew-symmetric invertible linear operator, i.e. $\mathcal{J}^* = -\mathcal{J}$.

Suppose that the functional $\mathcal{E}$ is invariant under a group $\mathcal{S}(\tau)$, with $\tau \in \mathbb{R}$, of unitary operators, so that $\mathcal{E}(\mathcal{S}(\tau)V) = \mathcal{E}(V)$ for all $V \in \mathcal{X}$ and $\tau \in \mathbb{R}$. Such a group of symmetries generates another conserved functional that we denote by $\mathcal{K}$. The functional $\mathcal{K}$ is given explicitly by

$$\mathcal{K}(V) = \langle \mathcal{J}^{-1}\mathcal{S}'(0)V, V\rangle$$

where $\mathcal{S}'(0)$ is the generator of the group $\mathcal{S}(\tau)$. Of interest are then solutions to (8.1) of the form $V(t) = \mathcal{S}(\omega t)Q$ for some fixed $Q \in \mathcal{X}$. Transforming equation (8.1) using $V(t) = \mathcal{S}(\omega t)U(t)$, we obtain

$$U_t = \mathcal{J}[\mathcal{E}'(U) - \omega\mathcal{K}'(U)] = \mathcal{J}\mathcal{H}'_\omega(U) \tag{8.2}$$

which is Hamiltonian with energy $\mathcal{H}_\omega(U) = \mathcal{E}(U) - \omega\mathcal{K}(U)$, where $\omega \in \mathbb{R}$ is a parameter. We seek stationary solutions $Q_\omega$ of (8.2), i.e. critical points of $\mathcal{H}_\omega$. Thus, throughout this section, we assume that $Q_\omega$ is a critical point of the energy $\mathcal{H}_\omega$, parametrized by $\omega$ in a certain interval. Note that (8.2) is still equivariant under the group $\mathcal{S}(\tau)$ so that we expect group orbits $\{\mathcal{S}(\tau)Q_\omega;\ \tau \in \mathbb{R}\}$ of stationary solutions for every fixed $\omega$. We assume that $\mathcal{S}'(0)Q_\omega \neq 0$ so that the group orbit is non-trivial.

As an example, consider the Korteweg–de Vries equation in Example 3. The symmetry group $\mathcal{S}(\tau)$ are the translations, $U(\cdot) \mapsto U(\cdot + \tau)$, and the parameter $\omega$ is the wave speed $c$: solutions $\mathcal{S}(\omega t)Q(\cdot)$ are travelling waves $Q(x - \omega t)$. In particular, they arise as families $Q(\cdot + \tau)$, parametrized by $\tau \in \mathbb{R}$, provided $Q(\cdot)$ is not a constant function.

The linearization of (8.2) about $Q_\omega$ is given by $\mathcal{L} = \mathcal{J}\mathcal{H}''_\omega(Q_\omega)$. We expect that the essential spectrum of $\mathcal{L}$ resides on the imaginary axis since $\mathcal{J}$ is skew-symmetric. In particular, the waves $Q_\omega$ will be at most marginally stable. Hence, to conclude nonlinear stability, we need to exploit the Hamiltonian nature of (8.2). If $\mathcal{H}''_\omega(Q_\omega)$ is positive definite[12] except for the simple eigenvalue $\lambda = 0$, then the wave $Q_\omega$ is nonlinearly stable because of conservation of energy: the Hamiltonian $\mathcal{H}_\omega$ serves as a Liapunov functional. In many important applications, however, the energy will not be definite. Instead, the Hessian $\mathcal{H}''_\omega(Q_\omega)$ has a unique simple eigenvalue with negative real part, so that the energy increases in all directions but one (not counting the neutral direction caused by symmetry). To compensate for this, we exploit that $\mathcal{K}$ is a conserved functional and restrict (8.2) to the invariant hyper-surface $\mathcal{C} = \{U \in \mathcal{X};\ \mathcal{K}(U) = \mathcal{K}(Q_\omega)\}$, with $\omega$ fixed. If $U = Q_\omega$ minimizes the energy $\mathcal{H}_\omega$ locally, subject to the constraint $U \in \mathcal{C}$, then the family $Q_\omega$ is nonlinearly stable.

It suffices therefore to find conditions that guarantee that $Q_\omega$ is a constrained minimizer of $\mathcal{H}_\omega$ for fixed $\omega$. Alternatively, we need that the Hessian $\mathcal{H}''_\omega(Q_\omega)$ restricted to the tangent space $\mathcal{K}'(Q_\omega)^\perp$ of $\mathcal{C}$ at $Q_\omega$ is positive definite except along the eigenfunction $\mathcal{S}'(0)Q_\omega$. This issue has been analyzed in [78, 79]: the criterion that guarantees that $Q_\omega$ is a constrained minimizer is that

$$\frac{\mathrm{d}}{\mathrm{d}\omega}[\mathcal{K}(Q_\omega)] = \langle \mathcal{H}''_\omega(Q_\omega)\partial_\omega Q_\omega, \partial_\omega Q_\omega\rangle < 0 \tag{8.3}$$

which, in fact, is equivalent to $d''(\omega) > 0$ where

$$d(\omega) = \mathcal{H}_\omega(Q_\omega).$$

In fact, a stronger result is true: $Q_\omega$ is nonlinearly stable if, and only if, the function $d(\omega)$ is convex (see [78, 79]). We refer to [78, Section 3] and [115, Section 2] for short proofs of the sufficiency of (8.3), and to [108, 114, 115, 116] for constrained minimization techniques and their relation to stability issues. Condition (8.3) decides upon definiteness of the Hessian of $\mathcal{H}_\omega$ restricted to the space $\mathcal{K}'(Q_\omega)^\perp$. It turns out that, if

[12] This statement is, of course, also true if $\mathcal{H}''_\omega(Q_\omega)$ is negative definite

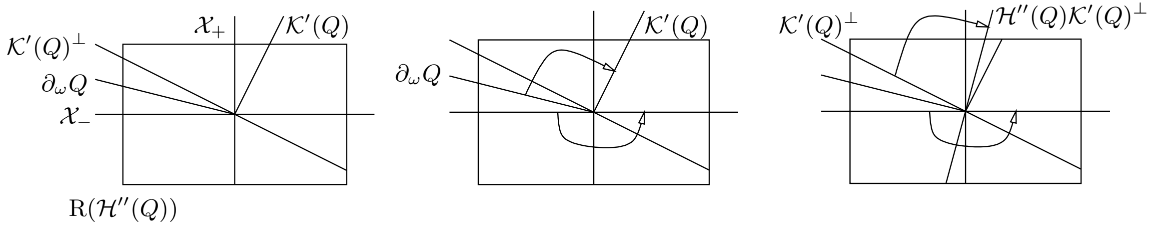


**Figure 15** Various subspaces of $\mathrm{R}(\mathcal{H}''_\omega(Q_\omega))$ are plotted in the left figure under the assumption that (8.3) is satisfied (the subscript $\omega$ is omitted in the plot). The subspaces $\mathcal{X}_-$ and $\mathcal{X}_+$ are the unstable and stable eigenspaces of $\mathcal{H}''_\omega(Q_\omega)$ on which this operator is negative and positive definite, respectively. The center plot illustrates that $\partial_\omega Q_\omega$ is mapped to $\mathcal{K}'(Q_\omega) = \mathcal{H}''_\omega(Q_\omega)\partial_\omega Q_\omega$ under $\mathcal{H}''_\omega(Q_\omega)$; this is true since $Q_\omega$ are critical points of $\mathcal{H}_\omega$ (the arrows indicate how $\mathcal{H}''_\omega(Q_\omega)$ acts on vectors). As a consequence, it is easily seen that the image of the space $\mathcal{K}'(Q_\omega)^\perp$ is as shown in the right figure. Thus, $\langle \mathcal{H}''_\omega(Q_\omega)U, U\rangle > 0$ for any $U \in \mathcal{K}'(Q_\omega)^\perp$.

(8.3) is met, then $Q_\omega$ is also a constrained minimizer of the original Hamiltonian $\mathcal{E}$. The geometric meaning of the criterion (8.3) in the plane is illustrated in Figure 15. If condition (8.3) is not met, then the wave is not a constrained minimizer; in fact, the linearization $\mathcal{L}$ about the wave has unstable eigenvalues (this statement requires an additional analysis for which we refer to [78, 79]).

In fact, analogous results are true for arbitrary finite-dimensional symmetry groups [79] and for general constrained minimization problems [115]; the parameter $\omega$ is then a finite-dimensional vector. As mentioned in Section 4.2.1, the criterion in (8.3) is related to the second-order derivative $D''(0)$ of the Evans function $D(\lambda)$. This relationship has been explored first in [134, 135] and has been put into an abstract framework in [21, 22] by exploiting a multi-symplectic formulation of the PDE, using symplectic operators for time and space. We refer also to [20] for a condition similar to (8.3), with $\omega \in \mathbb{R}^2$, that detects transverse instabilities in Hamiltonian PDEs with two-dimensional spatial variables.

Often, the spectrum of $\mathcal{H}''(Q)$ is relatively easy to compute. An interesting and important problem is then to infer as much as possible about the spectrum of the operator $\mathcal{L} = \mathcal{J}\mathcal{H}''(Q)$. We refer to [76, 77] for a very general approach to this problem and to [3, 109, 189, 190] for applications.

Note that the optimal stability result that one can hope for in the context of Hamiltonian systems is nonlinear orbital stability without asymptotic phase. In Hamiltonian PDEs on unbounded domains, it is sometimes possible to obtain asymptotic stability of travelling waves to Hamiltonian system by switching to a different norm that is not equivalent to the original one. This program has been carried out in [135] for the Korteweg–de Vries equation.

We remark that the stability of $\ell$-pulses in Hamiltonian PDEs is an interesting problem. Spectral stability can again be established using the results in Section 5. Nonlinear stability, however, is often far more complicated and, in general, unsolved. The reason is that, if we switch from the primary pulse to an $\ell$-pulse, there are $\ell$, rather than only one, negative eigenvalues of the operator $\mathcal{H}''(Q_\ell)$. To compensate for this, we would need to have $\ell$ independent conserved functionals $\mathcal{K}_j$ with associated parameters $\omega_j$, and the stability of an $\ell$-parameter family of solutions $Q_\omega$ can be concluded provided the $\ell \times \ell$ matrix with elements $\langle \mathcal{H}''_\omega(Q_\omega)\partial_{\omega_i}Q_\omega, \partial_{\omega_j}Q_\omega\rangle$ is negative definite. This program has been carried out in [115] for the multi-solitons of the Korteweg–de Vries equation: this is possible since the Korteweg–de Vries equation is integrable and admits infinitely-many independent conserved quantities. We also refer to [116] for a survey on constrained minimization and stability in Hamiltonian systems. If the equation has only finitely many independent conserved functionals (or if other conserved quantities are not known), then it is not clear how to proceed to establish nonlinear stability of multi-bump pulses.

#### Monotone (order-preserving) PDEs

Consider a reaction-diffusion system of the form

$$U_t = DU_{\xi\xi} + cU_\xi + F(U), \qquad \xi \in \mathbb{R}, \qquad U \in \mathbb{R}^N. \tag{8.4}$$

In many applications, e.g. to problems arising in combustion theory, the equation is monotone, i.e. the nonlinearity satisfies

$$\frac{\partial F_i}{\partial U_j}(U) > 0, \qquad i \neq j.$$

Under this assumption (in fact, under weaker conditions), the existence and stability of monotone fronts can often be proved. We refer to the monograph [182] and to [37, 141] for theory and applications. In fact, [182] is mainly concerned with results for equation (8.4) where the spatial variable $\xi$ lives on an unbounded cylinder, i.e. where $\xi \in \mathbb{R} \times \Omega$ for some bounded domain $\Omega \subset \mathbb{R}^m$.

#### Singularly perturbed reaction-diffusion systems

Singularly-perturbed reaction-diffusion systems of the form

$$\begin{aligned} \epsilon\tau u_t &= \epsilon^2 u_{xx} + f(u,v) \\ v_t &= \delta^2 v_{xx} + g(u,v) \end{aligned} \tag{8.5}$$

often allow for the construction, and the stability analysis, of travelling waves utilizing the equations in the singular limit as $\epsilon \to 0$.

Travelling waves can be constructed near the singular limit using geometric perturbation theory (see [91] for a recent survey) or using asymptotic matching (see [176, 112]). The stability of these travelling waves can be investigated using several different methods:

One possible approach is rigorous asymptotic matching (see again [112]). A second possibility is to use the Evans function [2]. In many cases, the elephant-trunk lemma [71] allows to write the Evans function for the full problem (8.5) as the product of the Evans functions to the slow and fast subsystems of (8.5) in the $\epsilon = 0$-limit. We refer to [15, 71, 142] for applications. Sometimes, the slow and fast system interact strongly, so that the Evans function is no longer computable as a product. In this situation, new and interesting phenomena occur, and we refer to [47] and [87] and the literature therein for further details. Using the approach from [47], the stability of spatially-periodic travelling waves that continue singular periodic waves has recently been investigated in [53] for the FitzHugh–Nagumo equation. Lastly, a different approach is the SLEP method introduced in [128, 129]. We refer to [127] for an extensive review of the SLEP method. The SLEP method and the approach via the Evans function are related [88].

# 9 Modulated, rotating, and travelling waves

We briefly report on extensions and generalizations of some of the theoretical results reviewed in the earlier sections.

#### Waves in heterogeneous media

Most of the results in this survey are concerned with homogeneous media, so that the underlying PDE has no explicit dependence on the spatial variables. Waves can also occur in heterogeneous media, and we refer to the recent survey [184].

### Travelling waves in cylindrical domains

We focused on travelling waves for a one-dimensional spatial variable. Often, however, one would be interested in parabolic equations

$$U_t = U_{xx} + \Delta U + F(U), \qquad (x, y) \in \mathbb{R} \times \Omega$$

on unbounded cylindrical domains with bounded, or unbounded, cross-section $\Omega \subset \mathbb{R}^m$. Here, $\Delta$ is the Laplace operator acting on the $y$-variable. In a moving frame $\xi = x - ct$, travelling waves become solutions $Q(\xi, y)$ to the elliptic problem

$$U_{\xi\xi} + \Delta U + cU_\xi + F(U) = 0, \qquad (\xi, y) \in \mathbb{R} \times \Omega.$$

The associated linearized eigenvalue-problem is given by

$$U_{\xi\xi} + \Delta U + cU_\xi + \partial_U F(Q(\xi, y))U = \lambda U.$$

We refer to [67, 84, 57, 182] and the references therein for various existence results. Methods that have been used to establish existence include the Conley index and spatial discretizations, the Leray-Schauder degree, and comparison principles (i.e. the construction of upper and lower solutions). Stability results can be found in [182] and in the comprehensive list of references therein.

Note that, using the reformulation

$$\begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta + \lambda - \partial_U F(Q(\xi, \cdot)) & -c \end{pmatrix} \begin{pmatrix} U \\ W \end{pmatrix}$$

as a first-order system, most of the methods and techniques reviewed in this article are also applicable to PDEs on cylindrical domains. We refer to [113, 137, 157] for details. This approach is based upon using the spatial variables in the unbounded directions as evolution variables. This concept, often referred to as spatial dynamics, has been introduced by Kirchgässner [101] to investigate small-amplitude solutions. We refer to [118, 120] and references therein for many subsequent articles where spatial dynamics has been utilized.

### Modulated waves

Modulated waves are solutions that are time-periodic in an appropriate moving coordinate frame, i.e. solutions $Q(x, t)$ that, for some wave speed $c$ and a certain temporal period $T$, satisfy

$$Q(x, t + T) = Q(x - cT, t), \qquad t, x \in \mathbb{R}.$$

Such waves may arise through Hopf bifurcations (when a pair of isolated complex-conjugate eigenvalues crosses the imaginary axis) or essential instabilities (when a part of the essential spectrum crosses the imaginary axis [152, 153, 158]). Another example are travelling waves in modulation equations such as the Ginzburg–Landau equation which often correspond to modulated waves of the PDE that has been reduced to the modulation equation (see e.g. [169, 179] for stability results in this context).

Most of the results presented in Sections 3-6 are also applicable to modulated waves, and we refer to [157] for details. The key is that exponential dichotomies, the main technical tool that we exploited, can also be constructed for linearizations about modulated waves.

As an example, consider again a reaction-diffusion system

$$U_t = DU_{\xi\xi} + cU_\xi + F(U), \qquad \xi \in \mathbb{R}$$

in an appropriate moving frame. The linearization about a modulated wave $Q(\xi,t)$ with $Q(\xi,t+T)=Q(\xi,t)$ is given by

$$U_t = DU_{\xi\xi} + cU_\xi + \partial_U F(Q(\xi,t))U.$$

Note that the coefficients of this equation are $T$-periodic in $t$. It turns out that the eigenvalues $\lambda$ of the $T$-map of the linearization can be characterized by exponential dichotomies to the first-order system

$$\begin{pmatrix} U_\xi \\ V_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha - \partial_U F(Q(\xi,\cdot))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix},$$

where $\lambda = \mathrm{e}^{\alpha T}$ and $(U,V)$ is $T$-periodic in $t$ for every $\xi \in \mathbb{R}$. The above system is of the same form as the equations that we studied in the earlier sections except that, for each fixed $\xi$, $(U,V)$ take values in a certain Hilbert space of $T$-periodic functions that depend on $t$ instead of in $\mathbb{C}^{2N}$. We refer to [157] for details.

#### Rotating waves in the plane

Consider the reaction-diffusion system

$$U_t = D\Delta U + F(U), \qquad U \in \mathbb{R}^N, \qquad x \in \mathbb{R}^2 \tag{9.1}$$

on the plane. A rotating wave is a solution $U(x,t)$ whose time-evolution is a rigid rotation with constant angular velocity $c$. Expressed in polar coordinates $(r,\varphi)$, a rotating wave is therefore of the form $U(r,\varphi,t) = Q(r,\varphi-ct)$. In a co-rotating coordinate frame, and using polar coordinates $(r,\varphi)$, equation (9.1) is given by

$$U_t = D\Delta_{r,\varphi} U + cU_\varphi + F(U), \qquad x \in \mathbb{R}^2. \tag{9.2}$$

A rotating wave with angular speed $c$ is a stationary solution to (9.2). Examples of rotating waves are Archimedean spiral waves that are stationary solutions $Q(r,\varphi)$ to (9.2) for an appropriate value of the angular velocity $c$ such that $Q(r,\varphi) \to Q_\infty(\kappa r + \varphi)$ as $r \to \infty$ for some $2\pi$-periodic function $Q_\infty(\psi)$. The function $Q_\infty(\psi)$ is a stationary wave-train solution to

$$U_t = D\kappa^2 U_{\psi\psi} + cU_\psi + F(U), \qquad \psi \in \mathbb{R}. \tag{9.3}$$

We refer to [121, 176] for background on spiral waves and various other waves in two and three space dimensions.

We cast (9.2) as a dynamical system in the radius $r$:

$$\begin{pmatrix} U_r \\ V_r \end{pmatrix} = \begin{pmatrix} V \\ -\left[\frac{V}{r} + \frac{U_{\varphi\varphi}}{r^2} + D^{-1}(cU_\varphi + F(U))\right] \end{pmatrix}.$$

Spiral waves $Q(r,\varphi)$ can then be thought of as fronts in the radial variable $r$ that connect the core state $Q(0,\varphi)$ at $r=0$ with the $r$-periodic wave train $Q_\infty(\kappa r+\varphi)$ as $r\to\infty$. We refer to [165] where this approach has been introduced to investigate Hopf bifurcations from homogeneous rest states to spiral waves with small amplitude.

To investigate the stability of spiral waves, consider the linearization of (9.2) about the spiral wave $Q$, written again as a first-order system in the radius $r$:

$$\begin{pmatrix} U_r \\ V_r \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\frac{\partial_{\varphi\varphi}}{r^2} - D^{-1}(c\partial_\varphi + \partial_U F(Q(r,\varphi)) - \lambda) & -\frac{1}{r} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{9.4}$$

This equation can again be investigated using exponential dichotomies (see [159]). Note that the limit of (9.4) as $r\to\infty$ is related to the linearization of (9.3) about the asymptotic wave train $Q_\infty$. In particular, the essential spectrum of the spiral can be computed using the essential spectrum of the asymptotic wave train $Q_\infty$.

For earlier results on the stability of spiral waves, we refer to [81] and to the review [176]. Various bifurcations of spiral waves to more complicated waves have been investigated in the literature, and we refer to [56, 58, 74, 160, 161] and the references therein for further details.

# References


[1] N. Akhmediev and A. Ankiewicz. *Solitons: Nonlinear pulses and beams*. Chapman and Hall, London, 1997.

[2] J.C. Alexander, R.A. Gardner, and C.K.R.T. Jones. A topological invariant arising in the stability analysis of travelling waves. *J. reine angew. Math.* **410** (1990) 167–212.

[3] J.C. Alexander, M. Grillakis, C.K.R.T. Jones, and B. Sandstede. Stability of pulses on optical fibers with phase-sensitive amplifiers. *Z. Angew. Math. Phys.* **48** (1997) 175–192.

[4] J.C. Alexander and C.K.R.T. Jones. Existence and stability of asymptotically oscillatory triple pulses. *Z. Angew. Math. Phys.* **44** (1993) 189–200.

[5] J.C. Alexander and C.K.R.T. Jones. Existence and stability of asymptotically oscillatory double pulses. *J. reine angew. Math.* **446** (1994) 49–79.

[6] J.C. Alexander and R.L. Sachs. Linear instability of solitary waves of a Boussinesq-type equation: A computer assisted computation. *Nonlinear World* **2** (1995) 471–507.

[7] E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen. LAPACK *Users' guide*. SIAM, Philadelphia, 1995.

[8] U.M. Ascher, R.M. Mattheij, and R.D. Russell. *Numerical solutions of boundary value problems for ordinary differential equations*. Prentice-Hall, Englewood Cliffs, 1988.

[9] P.W. Bates and C.K.R.T. Jones. Invariant manifolds for semilinear partial differential equations. *Dynamics Reported* **2** (1989) 1–38.

[10] T.B. Benjamin. The stability of solitary waves. *Proc. R. Soc. London A* **328** (1972) 153–183.

[11] W.-J. Beyn. Global bifurcations and their numerical computation. In *Continuation and bifurcations: Numerical techniques and applications*, pp. 169–181, D. Roose, A. Spence, and B. De Dier (Eds.). Kluwer, Doordrecht, 1990.

[12] W.-J. Beyn. The numerical computation of connecting orbits in dynamical systems. *IMA J. Numer. Anal.* **10** (1990) 379–405.

[13] W.-J. Beyn, A.R. Champneys, E. Doedel, W. Govaerts, Y.A. Kuznetsov, and B. Sandstede. Numerical continuation, and computation of normal forms. In *Handbook of dynamical systems III: Towards applications*, B. Fiedler, G. Iooss, and N. Kopell (Eds.). Elsevier.

[14] W.-J. Beyn and J. Lorenz. Stability of traveling waves: Dichotomies and eigenvalue conditions on finite intervals. *Numer. Funct. Anal. Opt.* **20** (1999) 201–244.

[15] A. Bose and C.K.R.T. Jones. Stability of the in-phase travelling wave solution in a pair of coupled nerve fibers. *Indiana Univ. Math. J.* **44** (1995) 189–220.

[16] L. Brevdo and T.J. Bridges. Absolute and convective instabilities of spatially periodic flows. *Phil. Trans. R. Soc. London A* **354** (1996) 1027–1064.

[17] J. Bricmont and A. Kupiainen. Renormalization group and the Ginzburg-Landau equation. *Comm. Math. Phys.* **150** (1992) 193–208.

[18] J. Bricmont and A. Kupiainen. Stability of moving fronts in the Ginzburg-Landau equation. *Comm. Math. Phys.* **159** (1994) 287–318.

[19] J. Bricmont, A. Kupiainen, and G. Lin. Renormalization group and asymptotics of solutions of nonlinear parabolic equations. *Comm. Pure Appl. Math.* **47** (1994) 893–922.

[20] T.J. Bridges. Universal geometric condition for the transverse instability of solitary waves. *Phys. Rev. Lett.* **84** (2000) 2614–2617.

[21] T.J. Bridges and G. Derks. Unstable eigenvalues and the linearization about solitary waves and fronts with symmetry. *Proc. R. Soc. London A* **455** (1999) 2427–2469.

[22] T.J. Bridges and G. Derks. The symplectic Evans matrix, and the instability of solitary waves and fronts with symmetry. Preprint.

[23] T.J. Bridges and A. Mielke. A proof of the Benjamin-Feir instability. *Arch. Rat. Mech. Anal.* **133** (1995) 145–198.

[24] R.J. Briggs. *Electron-steam interaction with plasmas.* MIT press, Cambridge, 1964.

[25] B. Buffoni, A.R. Champneys, and J.F. Toland. Bifurcation and coalescence of a plethora of homoclinic orbits for a Hamiltonian system. *J. Dynam. Diff. Eqns.* **8** (1996) 221–279.

[26] J. Carr and R. Pego. Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.* **42** (1989) 523–576.

[27] A.R. Champneys. Subsidiary homoclinic orbits to a saddle-focus for reversible systems. *Int. J. Bifurcation Chaos* **4** (1994) 1447–1482.

[28] A.R. Champneys and Y.A. Kuznetsov. Numerical detection and continuation of codimension-two homoclinic bifurcations. *Int. J. Bifurcation Chaos* **4** (1994) 795–822.

[29] A.R. Champneys, Y.A. Kuznetsov, and B. Sandstede. A numerical toolbox for homoclinic bifurcation analysis. *Int. J. Bifurcation Chaos* **6** (1996) 867–887.

[30] H.-C. Chang, E.A. Demekhin, and D.I. Kopelevich. Local stability theory of solitary pulses in an active medium. *Physica D* **97** (1996) 353–375.

[31] H.-C. Chang, E.A. Demekhin, and D.I. Kopelevich. Generation and suppression of radiation by solitary pulses. *SIAM J. Appl. Math.* **58** (1998) 1246–1277.

[32] S.-N. Chow, B. Deng, and B. Fiedler. Homoclinic bifurcation at resonant eigenvalues. *J. Dynam. Diff. Eqns.* **2** (1990) 177–244.

[33] S.-N. Chow and J.K. Hale. *Methods of bifurcation theory.* Springer, New York, 1982.

[34] E.A. Coddington and N. Levinson. *Theory of ordinary differential equations.* MacGraw-Hill, New York, 1955.

[35] P. Collet, J.-P. Eckmann, and H. Epstein. Diffusive repair for the Ginzburg-Landau equation. *Helv. Phys. Acta* **65** (1992) 56–92.

[36] W.A. Coppel. *Dichotomies in stability theory.* Lecture Notes in Mathematics **629**, Springer, Berlin, 1978.

[37] E.C.M. Crooks. Stability of travelling-wave solutions for reaction-diffusion-convection systems. Preprint.

[38] L. Debnath. *Nonlinear water waves.* Academic Press, Boston, 1994.

[39] B. Deng. The bifurcations of countable connections from a twisted heteroclinic loop. *SIAM J. Math. Anal.* **22** (1991) 653–679.

[40] B. Deng. The existence of infinitely many travelling front and back waves in the FitzHugh-Nagumo equations. *SIAM J. Math. Anal.* **22** (1991) 1631–1650.

[41] B. Deng. On Shilnikov's homoclinic-saddle-focus theorem. *J. Diff. Eqns.* **102** (1993) 305–329.

[42] R.L. Devaney. Homoclinic orbits in Hamiltonian systems. *J. Diff. Eqns.* **21** (1976) 431–438.

[43] F. Dias and C. Kharif. Nonlinear gravity and capillary-gravity waves. *Ann. Rev. Fluid. Mech.* **31** (1999) 301–346.

[44] E.J. Doedel, A.R. Champneys, T.F. Fairgrieve, Y.A. Kuznetsov, B. Sandstede, and X. Wang. AUTO97: Continuation and bifurcation software for ODEs (with HOMCONT). Concordia University, Technical Report, 1997.

[45] A. Doelman, R.A. Gardner, and T.J. Kaper. Stability analysis of singular patterns in the 1D Gray-Scott model: A matched asymptotics approach. *Physica D* **122** (1998) 1–36.

[46] A. Doelman, R.A. Gardner, and T.J. Kaper. A stability index analysis of 1-D patterns of the 1D Gray-Scott model. *Memoirs Amer. Math. Soc.* (to be published).

[47] A. Doelman, R.A. Gardner, and T.J. Kaper. Large stable pulse solutions in reaction-diffusion equations. *Indiana Univ. Math. J.* **49** (2000) (in press).

[48] U. Ebert and W. van Saarloos. Front propagation into unstable states: Universal algebraic convergence towards uniformly translating pulled fronts. *Physica D* (to be published).

[49] J.-P. Eckmann and C.E. Wayne. The nonlinear stability of front solutions for parabolic partial differential equations. *Comm. Math. Phys.* **161** (1994) 323–334.

[50] J.-P. Eckmann, C.E. Wayne, and P. Wittwer. Geometric stability analysis for periodic solutions of the Swift-Hohenberg equation. *Comm. Math. Phys.* **190** (1997) 173–211.

[51] S.-I. Ei. The motion of weakly interacting pulses in reaction-diffusion systems. Preprint.

[52] C. Elphick, E. Meron, and E.A. Spiegel. Patterns of propagating pulses. *SIAM J. Appl. Math.* **50** (1990) 490–503.

[53] E.G. Eszter. *An Evans function analysis of the stability of periodic travelling wave solutions of the FitzHugh-Nagumo system*. Ph.D. thesis, University of Massachusetts, Amherst, 1999.

[54] J. Evans. Nerve axon equations (iii): Stability of the nerve impulses. *Indiana Univ. Math. J.* **22** (1972) 577–594.

[55] B. Fiedler. Global pathfollowing of homoclinic orbits in two-parameter flows. In *Dynamics of nonlinear waves in dissipative systems: Reduction, bifurcation and stability*, pp. 79–145, G. Dangelmayr, B. Fiedler, K. Kirchgässner, and A. Mielke. Pitman Research Notes in Mathematics Series **352**, Longman, Harlow, 1996.

[56] B. Fiedler, B. Sandstede, A. Scheel, and C. Wulff. Bifurcations from relative equilibria of noncompact group actions: Skew products, meanders, and drifts. *Doc. Math.* **1** (1996) 479–505.

[57] B. Fiedler, A. Scheel, and M.I. Vishik. Large patterns of elliptic systems in infinite cylinders. *J. Math. Pures Appl.* **77** (1998) 879–907.

[58] B. Fiedler and D. Turaev. Normal forms, resonances, and meandering tip motions near relative equilibria of Euclidean group actions. *Arch. Rat. Mech. Anal.* **145** (1998) 129–159.

[59] P.C. Fife. Patterns formation in gradient systems. In *Handbook of dynamical systems III: Towards applications*, B. Fiedler, G. Iooss, and N. Kopell (Eds.). Elsevier.

[60] S. Focant and T. Gallay. Existence and stability of propagating fronts for an autocatalytic reaction-diffusion system. *Physica D* **120** (1998) 346–368.

[61] M.J. Friedman and E.J. Doedel. Numerical computation and continuation of invariant manifolds connecting fixed points. *SIAM J. Numer. Anal.* **28** (1991) 789–808.

[62] G. Fusco and J. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *J. Dynam. Diff. Eqns.* **1** (1989) 75–94.

[63] T. Gallay. Local stability of critical fronts in nonlinear parabolic partial differential equations. *Nonlinearity* **7** (1994) 741–764.

[64] T. Gallay and A. Mielke. Diffusive mixing of stable states in the Ginzburg-Landau equation. *Comm. Math. Phys.* **199** (1998) 71–97.

[65] T. Gallay and G. Raugel. Stability of travelling waves for a damped hyperbolic equation. *Z. Angew. Math. Phys.* **48** (1997) 451–479.

[66] T. Gallay and G. Raugel. Scaling variables and stability of hyperbolic fronts. *SIAM J. Math. Anal.* **32** (2000) 1–29.

[67] R.A. Gardner. Existence of multidimensional travelling wave solutions of an initial-boundary value problem. *J. Diff. Eqns.* **61** (1986) 335–379.

[68] R.A. Gardner. On the structure of the spectra of periodic travelling waves. *J. Math. Pures Appl.* **72** (1993) 415–439.

[69] R.A. Gardner. Spectral analysis of long wavelength periodic waves and applications. *J. reine angew. Math.* **491** (1997) 149–181.

[70] R.A. Gardner and C.K.R.T. Jones. Traveling waves of a perturbed diffusion equation arising in a phase field model. *Indiana Univ. Math. J.* **39** (1990) 1197–1222.

[71] R.A. Gardner and C.K.R.T. Jones. Stability of travelling wave solutions of diffusive predator-prey systems. *Trans. Amer. Math. Soc.* **327** (1991) 465–524.

[72] R.A. Gardner and K. Zumbrun. The gap lemma and geometric criteria for instability of viscous shock profiles. *Comm. Pure Appl. Math.* **51** (1998) 797–855.

[73] P. Glendinning. Subsidiary bifurcations near bifocal homoclinic orbits. *Math. Proc. Camb. Phil. Soc.* **105** (1989) 597–605.

[74] M. Golubitsky, V. LeBlanc, and I. Melbourne. Meandering of the spiral tip – an alternative approach. *J. Nonlinear Sci.* **7** (1997) 557–586.

[75] M. Golubitsky and D. Schaeffer. *Singularities and groups in bifurcation theory I.* Springer, Berlin, 1988.

[76] M. Grillakis. Linearized instability for nonlinear Schrödinger and Klein-Gordon equations. *Comm. Pure Appl. Math.* **41** (1988) 747–774.

[77] M. Grillakis. Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system. *Comm. Pure Appl. Math.* **43** (1990) 299–333.

[78] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry I. *J. Funct. Anal.* **74** (1987) 160–197.

[79] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry II. *J. Funct. Anal.* **94** (1990) 308–348.

[80] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Springer, New York, 1983.

[81] P.S. Hagan. Spiral waves in reaction-diffusion equations. *SIAM J. Appl. Math.* **42** (1982) 762–786.

[82] J. Härterich. Cascades of homoclinic orbits in reversible dynamical systems. *Physica D* **112** (1998) 187–200.

[83] P. Hartman. *Ordinary differential equations.* Birkhäuser, Boston, 1982.

[84] S. Heinze. *Traveling waves for semilinear parabolic partial differential equations in cylindrical domains.* Ph.D. thesis, University of Heidelberg, 1989.

[85] D. Henry. *Geometric theory of semilinear parabolic equations.* Lecture Notes in Mathematics **804**, Springer, New York, 1981.

[86] A.J. Homburg, H. Kokubu, and M. Krupa. The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic orbit. *Erg. Th. Dynam. Syst.* **14** (1994) 667–693.

[87] H. Ikeda and T. Ikeda. Bifurcation phenomena from standing pulse solutions in some reaction-diffusion systems. *J. Dynam. Diff. Eqns.* **12** (2000) 117–167

[88] H. Ikeda, Y. Nishiura, and Y. Suzuki. Stability of traveling waves and a relation between the Evans function and the SLEP equation. *J. reine angew. Math.* **475** (1996) 1–37.

[89] R.S. Johnson. *A modern introduction to the mathematical theory of water waves.* Cambridge University Press, Cambridge, 1997.

[90] C.K.R.T. Jones. Stability of the travelling wave solution of the FitzHugh-Nagumo system. *Trans. Amer. Math. Soc.* **286** (1984) 431–469.

[91] C.K.R.T. Jones. *Geometric singular perturbation theory.* C.I.M.E. Lectures, Montecatini Terme. Lecture Notes in Mathematics **1609**, Springer, Heidelberg, 1995.

[92] C.K.R.T. Jones, R.A. Gardner, and T. Kapitula. Stability of travelling waves for non-convex scalar viscous conservation laws. *Comm. Pure Appl. Math.* **46** (1993) 505–526.

[93] T. Kapitula. On the stability of travelling waves in weighted $L^\infty$ spaces. *J. Diff. Eqns.* **112** (1994) 179–215.

[94] T. Kapitula. Stability criterion for bright solitary waves of the perturbed cubic-quintic Schrödinger equation. *Physica D* **116** (1998) 95–120.

[95] T. Kapitula. The Evans function and generalized Melnikov integrals. *SIAM J. Math. Anal.* **30** (1999) 273–297.

[96] T. Kapitula and J. Rubin. Existence and stability of standing hole solutions to complex Ginzburg-Landau equations. *Nonlinearity* **13** (2000) 77–112.

[97] T. Kapitula and B. Sandstede. Stability of bright solitary wave solutions to perturbed nonlinear Schrödinger equations. *Physica D* **124** (1998) 58–103.

[98] T. Kapitula and B. Sandstede. A novel instability mechanism for bright solitary-wave solutions to the cubic-quintic Ginzburg-Landau equation. *J. Opt. Soc. Amer. B* **15** (1998) 2757–2762.

[99] R. Kapral and K. Showalter (Eds.). *Chemical waves and patterns.* Kluwer, Doordrecht, 1995.

[100] T. Kato. *Perturbation theory for linear operators.* Springer, New York, 1966.

[101] K. Kirchgässner. Wave-solutions of reversible systems and applications. *J. Diff. Eqns.* **45** (1982) 113–127.

[102] K. Kirchgässner. On the nonlinear dynamics of travelling fronts. *J. Diff. Eqns.* **96** (1992) 256–278.

[103] K. Kirchgässner and G. Raugel. Stability of fronts for a KPP-system: The non-critical case. In *Dynamics of nonlinear waves in dissipative systems: Reduction, bifurcation and stability*, pp. 147–208, G. Dangelmayr, B. Fiedler, K. Kirchgässner, and A. Mielke. Pitman Research Notes in Mathematics Series **352**, Longman, Harlow, 1996.

[104] M. Kisaka, H. Kokubu, and H. Oka. Bifurcations to $N$-homoclinic orbits and $N$-periodic orbits in vector fields. *J. Dynam. Diff. Eqns.* **5** (1993) 305–357.

[105] M. Krupa, B. Sandstede, and P. Szmolyan. Fast and slow waves in the FitzHugh-Nagumo equation. *J. Diff. Eqns.* **133** (1997) 49–97.

[106] J. Kutz and W. Kath. Stability of pulses in nonlinear optical fibers using phase-sensitive amplifiers. *SIAM J. Appl. Math.* **56** (1996) 611–626.

[107] Y.A. Kuznetsov. *Elements of applied bifurcation theory.* Springer, New York, 1995.

[108] S.P. Levandosky. A stability analysis of fifth-order water wave models. *Physica D* **125** (1999) 222–240.

[109] Y.A. Li and K. Promislow. Structural stability of non-ground state traveling waves of coupled nonlinear Schrödinger equations. *Physica D* **124** (1998) 137–165.

[110] Y.A. Li and K. Promislow. The mechanism of the polarizational mode instability in birefringent fiber optics. *SIAM J. Math. Anal.* **31** (2000) 1351–1373.

[111] X.-B. Lin. Using Melnikov's method to solve Shilnikov's problems. *Proc. R. Soc. Edinburgh A* **116** (1990) 295–325.

[112] X.-B. Lin. Construction and asymptotic stability of structurally stable internal layer solutions. *Trans. Amer. Math. Soc.* (to be published).

[113] G.J. Lord, D. Peterhof, B. Sandstede, and A. Scheel. Numerical computation of solitary waves in infinite cylindrical domains. *SIAM J. Numer. Anal.* **37** (2000) 1420–1454.

[114] J.H. Maddocks. Restricted quadratic forms and their application to bifurcation and stability in constrained variational principles. *SIAM J. Math. Anal.* **16** (1985) 47–68. Errata **19** (1988) 1256–1257.

[115] J.H. Maddocks and R.L. Sachs. On the stability of KdV multi-solitons. *Comm. Pure Appl. Math.* **46** (1993) 867–901.

[116] J.H. Maddocks and R.L. Sachs. Constrained variational principles and stability in Hamiltonian systems. In *Hamiltonian dynamical systems*, pp. 231–264. IMA Vol. Math. Anal. **63**, Springer, New York, 1995.

[117] E. Meron. Pattern formation in excitable media. *Physics Reports* **218** (1992) 1–66.

[118] A. Mielke. A spatial center manifold approach to steady bifurcations from spatially periodic patterns. In *Dynamics of nonlinear waves in dissipative systems: Reduction, bifurcation and stability*, pp. 209–262, G. Dangelmayr, B. Fiedler, K. Kirchgässner, and A. Mielke. Pitman Research Notes in Mathematics Series **352**, Longman, Harlow, 1996.

[119] A. Mielke. Instability and stability of rolls in the Swift-Hohenberg equation. *Comm. Math. Phys.* **189** (1997) 829–853.

[120] A. Mielke. The Ginzburg-Landau equation in its role as a modulation equation. In *Handbook of dynamical systems III: Towards applications*, B. Fiedler, G. Iooss, and N. Kopell (Eds.). Elsevier.

[121] J. Murray. *Mathematical biology.* Springer, Berlin, 1989.

[122] S. Nii. An extension of the stability index for travelling wave solutions and its application to bifurcations. *SIAM J. Math. Anal.* **28** (1997) 402–433.

[123] S. Nii. Stability of travelling multiple-front (multiple-back) wave solutions of the FitzHugh-Nagumo equations. *SIAM J. Math. Anal.* **28** (1997) 1094–1112.

[124] S. Nii. A topological proof of stability of N-front solutions of the FitzHugh-Nagumo equations. *J. Dynam. Diff. Eqns.* **11** (1999) 515–555.

[125] S. Nii. Accumulation of eigenvalues in a stability problem. *Physica D* **142** (2000) 70–86.

[126] S. Nii and B. Sandstede. On the connection between the geometry and stability of multi-bump pulses arising in homoclinic flip bifurcations. In preparation.

[127] Y. Nishiura. Coexistence of infinitely many stable solutions to reaction-diffusion systems in the singular limit. *Dynamics Reported* **3** (1994) 25–103.

[128] Y. Nishiura and H. Fujii. Stability of singularly perturbed solutions to systems of reaction-diffusion equations. *SIAM J. Math. Anal.* **18** (1987) 1726–1770.

[129] Y. Nishiura, M. Mimura, H. Ikeda, and H. Fujii. Singular limit analysis of stability of traveling wave solutions in bistable reaction-diffusion equations. *SIAM J. Appl. Math.* **49** (1990) 85–122.

[130] M. Or-Guil, I.G. Kevrekidis, and M. Bär. Stable bound states of pulses in an excitable medium. *Physica D* **135** (2000) 154–174.

[131] K.J. Palmer. Exponential dichotomies and transversal homoclinic points. *J. Diff. Eqns.* **55** (1984) 225–256.

[132] K.J. Palmer. Exponential dichotomies and Fredholm operators. *Proc. Amer. Math. Soc.* **104** (1988) 149–156.

[133] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer, Berlin, 1983.

[134] R. Pego and M. Weinstein. Eigenvalues, and instabilities of solitary waves. *Phil. Trans. R. Soc. London A* **340** (1992) 47–94.

[135] R. Pego and M. Weinstein. Asymptotic stability of solitary waves. *Comm. Math. Phys.* **164** (1994) 305–349.

[136] R. Pego and M. Weinstein. Convective linear stability of solitary waves for Boussinesq equations. *Stud. Appl. Math.* **99** (1997) 311–375.

[137] D. Peterhof, B. Sandstede, and A. Scheel. Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders. *J. Diff. Eqns.* **140** (1997) 266–308.

[138] J. Prüss. On the spectrum of $C^0$-semigroups. *Trans. Amer. Math. Soc.* **284** (1984) 847–857.

[139] G. Raugel and K. Kirchgässner. Stability of fronts for a KPP-system II: The critical case. *J. Diff. Eqns.* **146** (1998) 399–456.

[140] S.C. Reddy and L.N. Trefethen. Pseudospectra of the convection-diffusion operator. *SIAM J. Appl. Math.* **54** (1994) 1634–1649.

[141] J.-M. Roquejoffre, D. Terman, and V.A. Volpert. Global stability of traveling fronts and convergence towards stacked families of waves in monotone parabolic systems. *SIAM J. Math. Anal.* **27** (1996) 1261–1269.

[142] J.E. Rubin. Stability, bifurcations, and edge oscillations in standing pulse solutions to an inhomogeneous reaction-diffusion system. *Proc. R. Soc. Edinburgh A* **129** (1999) 1033–1079.

[143] B. Sandstede. *Verzweigungstheorie homokliner Verdopplungen*. Ph.D. thesis, University of Stuttgart, 1993.

[144] B. Sandstede. Convergence estimates for the numerical approximation of homoclinic solutions. *IMA J. Numer. Anal.* **17** (1997) 437–462.

[145] B. Sandstede. Instability of localized buckling modes in a one-dimensional strut model. *Phil. Trans. R. Soc. London A* **355** (1997) 2083–2097.

[146] B. Sandstede. Stability of multiple-pulse solutions. *Trans. Amer. Math. Soc.* **350** (1998) 429–472.

[147] B. Sandstede. Stability of N-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh-Nagumo equation. *SIAM J. Math. Anal.* **29** (1998) 183–207.

[148] B. Sandstede. Stability of multi-bump pulses in the complex cubic-quintic Ginzburg-Landau equation. In preparation.

[149] B. Sandstede. Weak interaction of pulses. In preparation.

[150] B. Sandstede, J.C. Alexander, and C.K.R.T. Jones. Existence and stability of $n$-pulses on optical fibers with phase-sensitive amplifiers. *Physica D* **106** (1997) 167–206.

[151] B. Sandstede, T. Kapitula and J. Kutz. Stability analysis of travelling waves in nonlocal equations and an application to modelocked pulses. Preprint.

[152] B. Sandstede and A. Scheel. Essential instability of pulses and bifurcations to modulated travelling waves. *Proc. R. Soc. Edinburgh A* **129** (1999) 1263–1290.

[153] B. Sandstede and A. Scheel. Spectral stability of modulated travelling waves bifurcating near essential instabilities. *Proc. R. Soc. Edinburgh A* **130** (2000) 419–448.

[154] B. Sandstede and A. Scheel. Gluing unstable fronts and backs together can produce stable pulses. *Nonlinearity* **13** (2000) 1465–1482.

[155] B. Sandstede, and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Physica D* (in press).

[156] B. Sandstede and A. Scheel. On the stability of periodic travelling waves with large spatial period. *J. Diff. Eqns.* (to be published).

[157] B. Sandstede and A. Scheel. On the structure of spectra of modulated travelling waves. *Math. Nachr.* (to be published).

[158] B. Sandstede and A. Scheel. Essential instabilities of fronts: Bifurcation, and bifurcation failure. *Dynam. Stabil. Syst.* (to be published).

[159] B. Sandstede and A. Scheel. Absolute versus convective instability of spiral waves. Preprint.

[160] B. Sandstede, A. Scheel, and C. Wulff. Dynamics of spiral waves on unbounded domains using center-manifold reductions. *J. Diff. Eqns.* **141** (1997) 122–149.

[161] B. Sandstede, A. Scheel, and C. Wulff. Bifurcations and dynamics of spiral waves. *J. Nonlinear Sci.* **9** (1999) 439–478.

[162] D.H. Sattinger. On the stability of waves of nonlinear parabolic systems. *Adv. Math.* **22** (1976) 312–355.

[163] B. Scarpellini. $L^2$-perturbations of periodic equilibria of reaction-diffusion systems. *Nonlinear Diff. Eqns. Appl.* **1** (1994) 281–311.

[164] B. Scarpellini. The principle of linearized instability for space-periodic equilibria of Navier-Stokes on an infinite plate. *Analysis* **15** (1995) 359–391.

[165] A. Scheel. Bifurcation to spiral waves in reaction-diffusion systems. *SIAM J. Math. Anal.* **29** (1998) 1399–1418.

[166] G. Schneider. Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation. *Comm. Math. Phys.* **178** (1996) 679–702.

[167] G. Schneider. Nonlinear stability of Taylor vortices in infinite cylinders. *Arch. Rat. Mech. Anal.* **144** (1998) 121–200.

[168] G. Schneider. Nonlinear diffusive stability of spatially periodic solutions— abstract theorem and higher space dimensions. In *Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997)*, pp. 159–167. Tohoku Math. Publ. **8**, 1998.

[169] G. Schneider. Existence and stability of modulating pulse-solutions in a phenomenological model of nonlinear optics. *Physica D* **140** (2000) 283–293.

[170] L.P. Shilnikov. A case of the existence of a countable number of periodic motions. *Soviet. Math. Dokl.* **6** (1965) 163–166.

[171] J. Smoller. *Shock waves and reaction-diffusion equations*. Springer, New York, 1994.

[172] Theme issue on "Localisation and solitary waves in solid mechanics". *Phil. Trans. R. Soc. London A* **355** (issue 1732) (1997).

[173] Theme issue on "Statics and dynamics of localisation phenomena". *Nonlinear Dynamics* **21** (issue 1) (2000).

[174] Theme issue on "Time-reversible symmetry in dynamical systems". *Physica D* **112** (issue 1-2) (1998).

[175] L.N. Trefethen. Pseudospectra of linear operators. *SIAM Rev.* **39** (1997) 383–406.

[176] J.J. Tyson and J.P. Keener. Singular perturbation theory for traveling waves in excitable media (a review). *Physica D* **32** (1988) 327–361.

[177] W. van Saarloos and P. Hohenberg. Fronts, pulses, sources, and sinks in the generalized complex Ginzburg-Landau equation. *Physica D* **56** (1992) 303–367.

[178] H. Uecker. Diffusive stability of rolls in the two-dimensional real and complex Swift-Hohenberg equation. *Comm. Partial Diff. Eqns.* **24** (1999) 2109–2146.

[179] H. Uecker. Stable modulating multi-pulse solutions for dissipative systems with a resonant spatially periodic forcing. *J. Nonlinear Sci.* (to be published).

[180] A. Vanderbauwhede. *Local bifurcation and symmetry*. Pitman Research Notes in Mathematics Series **75**, Pitman, Boston, 1982.

[181] A. Vanderbauwhede and B. Fiedler. Homoclinic period blow-up in reversible and conservative systems. *Z. Angew. Math. Phys.* **43** (1992) 292–318.

[182] A.I. Volpert, V.A. Volpert, and V.A. Volpert. *Traveling waves solutions of parabolic systems*. Transl. Math. Mono. **140**, Amer. Math. Soc., Providence, 1994.

[183] M.I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.* **16** (1985) 472–491.

[184] J. Xin. Front propagation in heterogeneous media. *SIAM Rev.* **42** (2000) 161–230.

[185] E. Yanagida. Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations. *J. Math. Biology* **22** (1985) 81–104.

[186] E. Yanagida. Branching of double pulse solutions from single pulse solutions in nerve axon equations. *J. Diff. Eqns.* **66** (1987) 243–262.

[187] E. Yanagida and K. Maginu. Stability of double-pulse solutions in nerve axon equations. *SIAM J. Appl. Math.* **49** (1989) 1158–1173.

[188] A.C. Yew. Multipulses of nonlinearly-coupled Schrödinger equations. *J. Diff. Eqns.* (to be published).

[189] A.C. Yew. Stability analysis of multipulses in nonlinearly-coupled Schrödinger equations. *Indiana Univ. Math. J.* **49** (2000) (in press).

[190] A.C. Yew, B. Sandstede, and C.K.R.T. Jones. Instability of multiple pulses in coupled nonlinear Schrödinger equations. *Phys. Rev. E* **61** (2000) 5886–5892.

[191] M.G. Zimmermann, S.O. Firle, M.A. Natiello, M. Hildebrand, M. Eiswirth, M. Bär, A.K. Bangia, and I.G. Kevrekidis. Pulse bifurcation and transition to spatiotemporal chaos in an excitable reaction-diffusion model. *Physica D* **110** (1997) 92–104.

[192] K. Zumbrun and P. Howard. Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.* **47** (1999) 741–872.