Homoclinic flip bifurcations in conservative reversible systems

Björn Sandstede

Abstract In this paper, flip bifurcations of homoclinic orbits in conservative reversible systems are analysed. In such systems, orbit-flip and inclination-flip bifurcations occur simultaneously. It is shown that multi-pulses either do not bifurcate at all at flip bifurcation points or else bifurcate simultaneously to both sides of the bifurcation point. An application to a fifth-order model of water waves is given to illustrate the results, and open problems regarding the PDE stability of multi-pulses are outlined.

1 Introduction

In this paper, we discuss flip bifurcations of homoclinic orbits in conservative reversible systems. Our main motivation for studying these bifurcations comes from the observation that spatially localized travelling waves of partial differential equations (PDEs) in one-dimensional extended domains can be found as homoclinic orbits of the underlying ordinary differential equation (ODE) that describes travelling waves. To illustrate this principle, consider the fifth-order PDE

$$u_t + \frac{2}{15}u_{xxxxx} - bu_{xxx} + 3uu_x + 2u_x u_{xx} + uu_{xxx} = 0, \qquad x \in \mathbb{R}, \tag{1}$$

which arises as the weakly nonlinear long-wave approximation to the classical gravity-capillary water-wave problem [2]. Here, $u(x,t)$ is the surface elevation measured with respect to the underlying normal water height, and the parameter b is the offset of the Bond number, which measures surface tension, from the value $\frac{1}{3}$. Travelling waves $u(x,t) = u(x+ct)$ of (1) satisfy the fourth-order equation

Björn Sandstede
Division of Applied Mathematics, Brown University, Providence, RI 02912, USA, e-mail: bjorn_sandstede@brown.edu

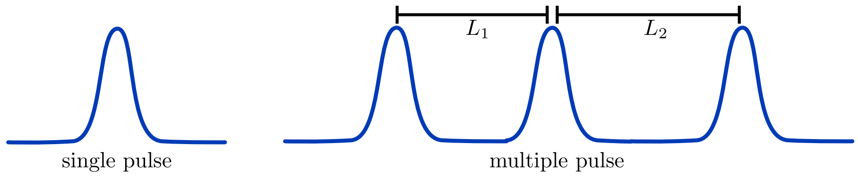

Fig. 1 The left panel illustrates the shape of a typical localized pulse $u(x)$: plotted is $u(x)$ vertically against the horizontal spatial variable x. Multiple pulses resemble several well-separated copies of a single pulse as indicated in the right panel for a 3-pulse, consisting of three copies. The distances L_1 and L_2 between consecutive pulses in the 3-pulse can be used to distinguish different multi-pulses.

$$\frac{2}{15}u^{\text{iv}} - bu'' + cu + \frac{3}{2}u^2 - \frac{1}{2}[u']^2 + [uu']' = 0, \tag{2}$$

where c denotes the wave speed. Localized wave profiles $u(x)$ of (1) that satisfy $\lim_{x\to\pm\infty} u(x) = 0$, which we will refer to as pulses, correspond therefore to homoclinic orbits of the first-order system obtained from the ODE (2).

Assume now that we found a pulse $u(x)$, which corresponds to a localized wave of elevation or suppression. In this case, it is of interest to see whether several copies of the pulse can be glued together to create a travelling pulse that consists of several regions of elevation or suppression as indicated in Figure 1. Several bifurcation scenarios are known at which multi-pulses of the form described above emerge, and we refer to [5] for a comprehensive survey. This paper focuses on homoclinic flip bifurcations, which come in two varieties. Orbit-flip bifurcations arise if the pulse is more localized than expected from the spatial eigenvalue structure of the equilibrium $u = 0$ of the ODE (2). Inclination-flip bifurcations, on the other hand, arise as follows: let $\mathscr{L}$ be the linearization of the PDE (1), formulated in a comoving frame, about the pulse, then this operator has an eigenvalue at the origin due to translation symmetry. Let $\psi(x)$ denote the associated eigenfunction of the adjoint operator $\mathscr{L}^*$; this eigenfunction has a natural interpretation as a solution of the adjoint variational equation of the ODE (2) about the pulse $u(x)$. An inclination-flip arises if the adjoint eigenfunction is more localized than expected.

Whether, and in what form, multi-pulses bifurcate at an orbit- or inclination flip bifurcation depends strongly on whether the underlying travelling-wave system (2) has additional structure. Here, we focus on two possible structures that commonly arise. The first structure is equivariance of (2) under the reflection $x \mapsto -x$, which we will refer to as reversibility. The second relevant structure is whether the travelling-wave system admits a first integral, that is, a real-valued quantity H that does not change when evaluated along solutions: we refer to such systems as conservative.

It was shown in [12] that orbit-flip bifurcations of nonconservative reversible systems lead to N-pulses for each N. A similar result was shown in [13] for nonreversible conservative systems. It turns out that the water-wave problem (2) is both reversible and conservative (we will show this in §4). Neither of the aforementioned results therefore applies to (2), and this paper focuses on deriving bifurcation results for this case. As we will see, the results for reversible conservative systems are quite different from those for systems that admit one but not both of these structures.

The main open issue is the stability of the multi-pulses found in this and other bifurcation scenarios with respect to the underlying partial differential equation. The fifth-order model given above is a Hamiltonian PDE, and stability for such equations is subtle. We will comment in detail on the outstanding issues in the conclusions section at the end of this paper.

This paper is structured as follows. The precise setting and the main results are formulated in §2. Our results are proved in §3, and we consider the application to the water-wave problem in §4. Conclusions and open problems are presented in §5.

2 Main results

In this section, we state the setting, assumptions, and main results more formally. We consider the ordinary differential equation

$$u' = f(u,\mu), \qquad (u,\mu) \in \mathbb{R}^{2n} \times \mathbb{R}, \tag{3}$$

where f is a smooth nonlinearity. We assume that (3) is reversible and conservative in the following sense.

Hypothesis (H1) (Reversibility) *There exists a linear map* $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *such that* $R^2 = \mathrm{id}$*, the fixed-point space* $\mathrm{Fix}(R)$ *of the reverser* R *satisfies* $\dim \mathrm{Fix}(R) = n$*, and* $Rf(u,\mu) = -f(Ru,\mu)$ *for all* $(u,\mu) \in \mathbb{R}^{2n} \times \mathbb{R}$.

We call a solution $u(x)$ reversible or symmetric with respect to the reverser R if $u(0) \in \mathrm{Fix}(R)$. Any symmetric solution u automatically satisfies

$$u(-x) = Ru(x), \qquad x \in \mathbb{R}.$$

We also assume the (3) is conservative, that is, that it admits a conserved quantity or first integral that is compatible with the reverser R.

Hypothesis (H2) (Conservative system) *There exists a smooth function* $H : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ *such that* $H_u(u,\mu) f(u,\mu) = 0$ *for all* $(u,\mu) \in \mathbb{R}^{2n} \times \mathbb{R}$*, and* $H_u(u,\mu) = 0$ *only at a discrete set of points in* $\mathbb{R}^{2n}$ *for each fixed* $\mu \in \mathbb{R}$*. Furthermore, we assume that* H *is invariant under the reverser* R*, that is,* $H(Ru,\mu) = H(u,\mu)$ *for all* (u,μ).

If Hypothesis (H2) is met, then $H(u(x),\mu) = H(u(0),\mu)$ along any solution $u(x)$ of (3). Hamiltonian systems given by

$$u' = JH_u(u,\mu), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad u \in \mathbb{R}^n \times \mathbb{R}^n$$

are a particular example of conservative systems.

Throughout, we assume that zero is a hyperbolic equilibrium of (3) for all μ near zero so that $f(0,\mu)=0$ and $f_u(0,\mu)$ is hyperbolic for μ near zero. We also assume that (3) admits a reversible homoclinic solution $h_0(x)$ to the origin for $\mu=0$.

Hypothesis (H3) *There is a solution $h_0(x)$ of (3) for $\mu=0$ with $h_0\not\equiv 0$ such that*

(i) $\lim_{x\to\pm\infty} h_0(x)=0$*;*
(ii) $T_{h_0(0)}W^{\mathrm{s}}(0)\cap T_{h_0(0)}W^{\mathrm{u}}(0)=\mathbb{R}h_0'(0)$*;*
(iii) $h_0(0)\in \mathrm{Fix}(R)$.

Hypotheses (H1) and (H2) each imply that the spectrum of $f_u(0,\mu)$ is symmetric with respect to the imaginary axis; see, for instance, [5, 14]. We assume that the leading eigenvalues of the origin are real.

Hypothesis (H4) *The spectrum of the equilibrium $u=0$ is given by*

$$\mathrm{spec}(f_u(0,\mu))=\sigma^{\mathrm{s}}\cup\{\pm\lambda^{\mathrm{u}}(\mu),\pm\lambda^{\mathrm{uu}}(\mu)\}\cup\sigma^{\mathrm{u}}$$

where $\pm\lambda^{\mathrm{u}}(\mu)$ and $\pm\lambda^{\mathrm{uu}}(\mu)$ are simple eigenvalues with $0<\lambda^{\mathrm{u}}(\mu)<\lambda^{\mathrm{uu}}(\mu)$. We also assume that there is a constant λ^{r} with $\lambda^{\mathrm{r}}>\lambda^{\mathrm{uu}}(\mu)$ so that $\mathrm{Re}\,\sigma^{\mathrm{s}}<-\lambda^{\mathrm{r}}$ and $\mathrm{Re}\,\sigma^{\mathrm{u}}>\lambda^{\mathrm{r}}$ for all μ near zero.

Hypothesis (H3) implies that there exists a smooth one-parameter family $h_\mu(x)$ of homoclinic solutions for μ close to zero that all satisfy Hypothesis (H3); see again [5, 14], for instance. We assume that these homoclinic orbits undergo an orbit-flip bifurcation at $\mu=0$:

Hypothesis (H5) *We assume that $h_0(x)\in W^{\mathrm{uu}}(0)$ for $\mu=0$ and that*

(i) $\lim\limits_{x\to-\infty}\mathrm{e}^{-\lambda^{\mathrm{uu}}x}h_0(x)=v_{\mathrm{uu}}\neq 0$*;*
(ii) $\lim\limits_{x\to-\infty}\mathrm{e}^{-\lambda^{\mathrm{u}}x}\frac{\mathrm{d}}{\mathrm{d}\mu}h_\mu(x)|_{\mu=0}=v_{\mathrm{u}}\neq 0.$

It follows that v_{u} and v_{uu} are eigenvectors of $f_u(0,0)$ that belong to the eigenvalues $\lambda^{\mathrm{u}}(0)$ and $\lambda^{\mathrm{uu}}(0)$, respectively. The quantities

$$b^j:=\langle H_{uu}(0,0)v_j,Rv_j\rangle=H_{uu}(0,0)[v_j,Rv_j],\qquad j=\mathrm{u},\mathrm{uu} \tag{4}$$

will play an important role in our result. The sign of the product $b^{\mathrm{u}}b^{\mathrm{uu}}$ has the following geometric interpretation. First, using that H cannot change along the stable or unstable manifolds of the origin, we can show that

$$b^j=\langle H_{uu}(0,0)v_j,Rv_j\rangle=\langle H_{uu}(0,0)(v_j+Rv_j),(v_j+Rv_j)\rangle,\qquad j=\mathrm{u},\mathrm{uu}.$$

Thus, b^{u} measures how the energy changes if we move along the direction $v_{\mathrm{u}}+Rv_{\mathrm{u}}$, which is the spine of the cone formed by the eigenvectors v_{u} and Rv_{u} belonging to the eigenvalues λ^{u} and $-\lambda^{\mathrm{u}}$, respectively, of $f_u(0,0)$. In particular, $b^{\mathrm{u}}>0$ indicates that the energy increases in this direction, while $b^{\mathrm{u}}<0$ means the energy decreases. The quantity b^{uu} has same interpretation for the cone in the strong stable and unstable eigenspace. The product $b^{\mathrm{u}}b^{\mathrm{uu}}$ is therefore positive if the energy increases or decreases in both cones, while $b^{\mathrm{u}}b^{\mathrm{uu}} < 0$ means that the energy increases in one cone and decreases in the other cone. In order to be able to pass from $h_\mu(L)$ to $h_\mu(-L)$, we need to pass through the product of these cones near the equilibrium. Since energy is conserved, this seems possible only if the zero energy level set intersects these cones: this happens only if $b^{\mathrm{u}}b^{\mathrm{uu}} < 0$. Thus, we expect N-pulses to exist only when $b^{\mathrm{u}}b^{\mathrm{uu}}$ is negative, but not if $b^{\mathrm{u}}b^{\mathrm{uu}}$ is positive. Our main result confirms this intuition.

Theorem 1. *Assume that Hypotheses (H1)–(H5) are met. For each $N > 1$, there exist numbers $\mu_N > 0$ and $L_N \gg 1$ with the following properties.*

(i) If $b^{\mathrm{u}}b^{\mathrm{uu}} > 0$, then (3) with $|\mu| < \mu_N$ does not admit a homoclinic orbit that makes N distinct loops near the primary orbit $h_0(x)$, where each return time is larger than L_N.

(ii) If $b^{\mathrm{u}}b^{\mathrm{uu}} < 0$, then (3) has, for each $\mu \neq 0$ with $|\mu| < \mu_N$, a unique homoclinic orbit that makes N distinct loops near the primary orbit $h_0(x)$, where each return time is larger L_N. This orbit is reversible, and the return times between consecutive pulses are given, to leading order, by

$$L = \frac{\ln|\mu|}{\lambda^{\mathrm{u}} - \lambda^{\mathrm{uu}}} + L_*$$

for some constant $L_* \in \mathbb{R}$.

In other words, N-pulses either do not emerge at all or else emerge to either side of $\mu = 0$. This is in contrast to many other homoclinic flip bifurcations, where solutions bifurcate either sub- or super-critically. In particular, N-pulses bifurcate to one side only at orbit-flip bifurcations in non-conservative reversible systems [12] and in non-reversible conservative systems [13].

3 Proof of Theorem 1

We apply Lin's method [8] to prove the existence and non-existence of the N-pulses near a homoclinic flip bifurcation. This method is explained in detail in [12], see also [5], and we shall follow here the same strategy and use the same notation as in [12].

Before we can state the results from [8, 10, 12], we introduce additional notation. Recall that we assumed that our system (3) is conservative. As shown for instance in [14], this property implies that the functions

$$\psi_\mu(x) = \nabla_u H(h_\mu(x), \mu) \tag{5}$$

are nontrivial bounded solutions to the adjoint variational equations

$$w' = -f_u(h_\mu(x), \mu)^* w \tag{6}$$

associated with the homoclinic orbits $h_\mu(x)$. We can now state the results from [8, 10, 12] that express conditions for the existence of N-pulses. Fix any natural number $N > 1$, then the results in [8, 10, 12] state that there are numbers μ_* and L_* such equation (3) with $|\mu| < \mu_*$ has an N-homoclinic orbit that is pointwise close to the orbit of $h_0(x)$ and follows $h_0(x)$ N-times if, and only if, the equations

$$\langle \psi_\mu(-L_{j-1}), h_\mu(L_{j-1})\rangle - \langle \psi_\mu(L_j), h_\mu(-L_j)\rangle + R_j(L,\mu) = 0 \tag{7}$$

with $j = 1,\dots,N-1$ has a solution $L = (L_j)_{j=1,\dots,N-1}$ with $L_j \geq L_*$ and $L_0 = \infty$. The numbers L_j are the return times of consecutive homoclinic loops to a fixed section at $h_0(0)$ or, alternatively, the distances between consecutive pulses in the corresponding multi-pulse. The functions $R_j(L,\mu)$ are higher-order terms, which we will estimate below. In restricting the index j in (7) to the set $j = 1,\dots,N-1$, we have used [12, Lemma 3.2] which asserts that the Nth equation will be satisfied automatically due to the energy constraint H provided the first $N-1$ equations are met.

Before stating the estimates for the remainder terms $R_j(L,\mu)$, we simplify (7) further. Reversibility of $h_\mu(x)$ and compatibility of R and H imply that

$$\begin{aligned} h_\mu(x) &= Rh_\mu(-x) \\ \psi_\mu(-x) &= \nabla_u H(h_\mu(-x),\mu) = \nabla_u H(Rh_\mu(x),\mu) = R^*\nabla_u H(h_\mu(x),\mu) = R^*\psi_\mu(x). \end{aligned}$$

Thus, (7) can be written as

$$\langle \psi_\mu(L_{j-1}), h_\mu(-L_{j-1})\rangle - \langle \psi_\mu(L_j), h_\mu(-L_j)\rangle + R_j(L,\mu) = 0, \quad j = 1,\dots,N-1.$$

Using that $L_0 = \infty$, we can recursively add the $(j-1)$th equation to the jth equation to obtain the new equivalent system

$$\langle \psi_\mu(L_j), h_\mu(-L_j)\rangle - R_j(L,\mu) - R_{j-1}(L,\mu) = 0, \quad j = 1,\dots,N-1$$

or, equivalently,

$$\langle \nabla_u H(h_\mu(L_j),\mu), h_\mu(-L_j)\rangle - R_j(L,\mu) - R_{j-1}(L,\mu) = 0, \quad j = 1,\dots,N-1 \tag{8}$$

with $R_0 \equiv 0$.

Next, we derive expressions for the scalar product that appears in (8). Since we assumed that $f_u(0,\mu)$ is hyperbolic, and therefore in particular invertible, it follows easily from differentiating $H_u(u,\mu)f(u,\mu)$ with respect to x that $H_u(0,\mu) = 0$. Using this property together with [12, (3.8)] and (5), we obtain the expansions

$$\begin{aligned} h_\mu(-x) &= \mu \mathrm{e}^{-\lambda^{\mathrm{u}} x} v_{\mathrm{u}} + \mathrm{e}^{-\lambda^{\mathrm{uu}} x} v_{\mathrm{uu}} + v_{\mathrm{r}}(x) \\ &\quad + \mathrm{O}\left(|\mu|(|\mu|\mathrm{e}^{-\lambda^{\mathrm{u}} x} + \mathrm{e}^{-2\lambda^{\mathrm{u}} x} + \mathrm{e}^{-\lambda^{\mathrm{uu}} x}) + \mathrm{e}^{-2\lambda^{\mathrm{uu}} x}\right) \\ \nabla_u H(h_\mu(x),\mu) &= H_{uu}(0,\mu)h_\mu(x) + \mathrm{O}(|h_\mu(x)|^2) \\ &= H_{uu}(0,\mu)Rh_\mu(-x) + \mathrm{O}(|h_\mu(-x)|^2) \end{aligned} \tag{9}$$

in the limit $x \to \infty$, where the function $v_{\mathrm{r}}(x)$ lies in the eigenspace associated with σ^{u} and decays faster than $\mathrm{e}^{-\lambda^{\mathrm{r}} x}$ as $x \to \infty$. Using these expansions, recalling the definition (4) of the quantities b^j, and using the fact that $H_{uu}(0,0)v$ is an eigenvector of $f_u(0,0)^*$ belonging to the eigenvalue $-\lambda$ whenever v is an eigenvector of $f_u(0,0)$ belonging to the eigenvalue λ, a straightforward calculation shows that

$$\begin{aligned} \langle \nabla_u H(h_\mu(x),\mu), h_\mu(-x)\rangle &= \mu^2 b^{\mathrm{u}} \mathrm{e}^{-2\lambda^{\mathrm{u}} x} + b^{\mathrm{uu}} \mathrm{e}^{-2\lambda^{\mathrm{uu}} x} \\ &\quad + \mathrm{O}\Big(\mathrm{e}^{-2\lambda^{\mathrm{r}} x} + |\mu| [\mathrm{e}^{-(2\lambda^{\mathrm{u}}+\lambda^{\mathrm{uu}})x} + \mathrm{e}^{-2\lambda^{\mathrm{uu}} x}] \\ &\quad + |\mu|^2 [\mathrm{e}^{-(\lambda^{\mathrm{u}}+\lambda^{\mathrm{uu}})x} + \mathrm{e}^{-3\lambda^{\mathrm{u}} x}] + |\mu|^3 \mathrm{e}^{-2\lambda^{\mathrm{u}} x}\Big) \end{aligned} \tag{10}$$

uniformly in μ near zero and $x \gg 1$. It remains to derive estimates on the remainder terms R_j.

Lemma 1. *Under the hypotheses of Theorem 1, the error terms $R_j(L,\mu)$ satisfy*

$$R_j(L,\mu) = \mathrm{O}\left(\left(\mathrm{e}^{-\lambda^{\mathrm{u}} L_{j-1}} + \mathrm{e}^{-\lambda^{\mathrm{u}} L_j}\right) \sum_{k=1}^{N-1} \left(\mu^2 \mathrm{e}^{-2\lambda^{\mathrm{u}} L_k} + \mathrm{e}^{-2\lambda^{\mathrm{uu}} L_k}\right)\right), \tag{11}$$

and the error terms can be differentiated.

Proof. The estimates given in [12, Theorem 3] are not sufficient to get the statement of the theorem. We therefore return to [10, §3.3.2] where the relevant expression for the bifurcation equations are recorded on [10, top of page 99]. The integral terms appearing in [10, top of page 99] can be estimated by

$$\left(\mathrm{e}^{-\lambda^{\mathrm{u}} L_{j-1}} + \mathrm{e}^{-\lambda^{\mathrm{u}} L_j}\right) \sum_{k=1}^{N-1} \left(\mu^2 \mathrm{e}^{-2\lambda^{\mathrm{u}} L_k} + \mathrm{e}^{-2\lambda^{\mathrm{uu}} L_k}\right)$$

using [10, Lemma 3.20]. The scalar products appearing in [10, top of page 99] are given by

$$\langle \psi_\mu(-L_j), h_\mu(L_j)\rangle + \mathrm{O}\left(\left(\mathrm{e}^{-\lambda^{\mathrm{u}} L_{j-1}} + \mathrm{e}^{-\lambda^{\mathrm{u}} L_j}\right) \sum_{k=1}^{N-1} \left(\mu^2 \mathrm{e}^{-2\lambda^{\mathrm{u}} L_k} + \mathrm{e}^{-2\lambda^{\mathrm{uu}} L_k}\right)\right)$$

once [10, (3.42), (3.43) and (3.39)] are used, which completes the proof. □

As in [10, 12], we replace the variables L_j and μ by the new variables

$$\mu = \pm r^{\beta/2}, \qquad a_j r = \mathrm{e}^{-2\lambda^{\mathrm{u}} L_j} \tag{12}$$

for $a_j > 0$ and $r \geq 0$, where the exponent $\beta > 0$ will be chosen later. We also define

$$\alpha = \frac{\lambda^{\mathrm{uu}}}{\lambda^{\mathrm{u}}} - 1 = \frac{\lambda^{\mathrm{uu}} - \lambda^{\mathrm{u}}}{\lambda^{\mathrm{u}}}.$$

Substituting the expansion (10) and the estimates (11) into the bifurcation equations (8) and rewrite them using (12), we arrive, after some tedious but straightforward manipulations, at the system

$$a_j r^{1+\beta} b^{\mathrm{u}} + a_j^{1+\alpha} r^{1+\alpha} b^{\mathrm{uu}} + \mathrm{O}\left(\max\left\{r^{1+\alpha+\gamma}, r^{1+\beta+\gamma}\right\}\right) = 0 \tag{13}$$

where $j = 1, \ldots, N-1$. Here, $\gamma > 0$ is a positive constant that depends only on the quantities λ^{u}, λ^{uu}, and λ^{r} but not on L or μ. Observe that nontrivial solutions of (13) can exist only in the scaling $\alpha = \beta$ for which (13) becomes

$$a_j r^{1+\alpha} b^{\mathrm{u}} + a_j^{1+\alpha} r^{1+\alpha} b^{\mathrm{uu}} + \mathrm{O}(r^{1+\alpha+\gamma}) = 0, \qquad j = 1, \ldots, N-1.$$

Dividing by $r^{1+\alpha}$, we obtain

$$a_j \left(b^{\mathrm{u}} + a_j^{\alpha} b^{\mathrm{uu}}\right) + \mathrm{O}(r^{\gamma}) = 0, \qquad j = 1, \ldots, N-1. \tag{14}$$

If $b^{\mathrm{u}} b^{\mathrm{uu}} > 0$, it is not difficult to see that (14) cannot have any solutions other than $a_j = 0$ for all j which corresponds to the persisting homoclinic orbit $h_\mu(x)$.

Thus, let us assume from now on that $b^{\mathrm{u}} b^{\mathrm{uu}} < 0$. In this case, (14) has the positive solution

$$a_j^* = \left(-\frac{b^{\mathrm{u}}}{b^{\mathrm{uu}}}\right)^{1/\alpha} > 0, \qquad j = 1, \ldots, N-1 \tag{15}$$

at $r = 0$, and we can solve (14) near this solution for $r > 0$ by the implicit function theorem.

This completes the existence part of Theorem 1. The obtained N-homoclinic orbit is reversible since we could simply have solved the equations for $j = 1, \ldots, [N/2]$, with $[x]$ being the largest integer smaller than x, and setting $a_{N-j} := a_j$ for $j = 1, ..., [N/2]$. Applying [12, Lemma 3.1] then shows that any solution to this truncated system corresponds to a reversible N-homoclinic orbit of (3). Proceeding as above though, we find the same solution (15), which therefore must be symmetric. Lastly, the uniqueness of the N-homoclinic orbits can be proved as in [12, Lemma 3.6].

4 Application to a fifth-order model for water waves

We now apply Theorem 1 to the equation (1), now written as

$$u_t + \partial_x\left(\frac{2}{15}u_{xxxx} - bu_{xx} + \frac{3}{2}u^2 + \frac{1}{2}u_x^2 + uu_{xx}\right) = 0, \qquad x \in \mathbb{R}, \tag{16}$$

which, as mentioned in the introduction, arises as a long-wave approximation to the gravity-capillary water-wave problem [2]. Localized travelling waves $u(x,t) = u(x+ct)$ of (16) satisfy the fourth-order equation

$$\frac{2}{15}u^{\mathrm{iv}} - bu'' + cu + \frac{3}{2}u^2 + \frac{1}{2}[u']^2 + uu'' = 0. \tag{17}$$

Note that (17) is reversible under the reflection $x \mapsto -x$. This equation is also Hamiltonian, and hence conservative: indeed, as shown in [2], the variables

$$q_1 = u, \quad q_2 = u', \quad p_1 = -\frac{2}{15}u''' + bu' - uu', \quad p_2 = \frac{2}{15}u''$$

make (17) Hamiltonian with respect to the energy

$$H = -\frac{1}{2}q_1^3 - \frac{c}{2}q_1^2 + p_1 q_2 - \frac{b}{2}q_2^2 + \frac{15}{4}p_2^2 + \frac{1}{2}q_1 q_2^2$$

and the symplectic operator J

$$J: \quad (q_1, q_2, p_1, p_2) \longmapsto (p_1, p_2, -q_1, -q_2).$$

In these coordinates, the reverser R becomes

$$R: \quad (q_1, q_2, p_1, p_2) \longmapsto (q_1, -q_2, -p_1, p_2),$$

and we have $H \circ R = H$ as required. In particular, Hypotheses (H1) and (H2) are met.

As shown in [2, (4.3)], equation (17) has the explicit localized solution

$$u_*(x) = 3\left(b + \frac{1}{2}\right)\operatorname{sech}^2\left(\sqrt{3(2b+1)}\,\frac{x}{2}\right) \tag{18}$$

for $b > -\frac{1}{2}$, with wave speed given by

$$c = c_*(b) = \frac{3}{5}(2b+1)(b-2). \tag{19}$$

Linearizing (17) about $u = 0$, we find that its eigenvalues satisfy the relation

$$\frac{2}{15}\lambda^4 - b\lambda^2 + c = 0. \tag{20}$$

Substituting $c = c_*(b)$ from (19) and computing the unstable spatial eigenvalues, we find that they are given by

$$\lambda^{\mathrm{u}} = \frac{1}{2}\sqrt{6(b-2)}, \qquad \lambda^{\mathrm{uu}} = \sqrt{3(2b+1)}. \tag{21}$$

In particular, the equilibrium $u = 0$ is hyperbolic when $b > 2$, and this will be the region we shall focus on from now on. We are interested in applying Theorem 1 to the system (17), where the speed c, varied near $c = c_*(b)$, plays the role of the parameter μ that appears in §2. We now discuss the validity of the Hypotheses required for Theorem 1 to hold.

Comparing with (17), we see that the homoclinic orbit u_* is indeed in an orbit-flip configuration for all such b, and we see that Hypotheses (H3)(i)+(iii), (H4), and (H5)(i) are satisfied.

Thus, it remains to discuss Hypotheses (H3)(ii) and (H5)(ii): we do not have an analytical proof of their validity but describe now how they can be checked numerically. Restated in a more convenient formulation, Hypothesis (H3)(ii) assumes that $u_*'(x)$ is the only bounded solution of the variational equation

$$\mathscr{L}v := \frac{2}{15}v^{\mathrm{iv}} - bv'' + c_*(b)v + 3u_*(x)v + u_*'(x)v' + u_*(x)v'' + vu_*''(x) = 0. \qquad (22)$$

In other words, this hypothesis requires that the eigenvalue $\lambda_{\mathrm{pde}} = 0$ of the operator $\mathscr{L}$ posed on $L^2(\mathbb{R})$ is simple. Discretizing the derivatives in the operator $\mathscr{L}$ by centered finite differences and calculating its spectrum numerically in MATLAB using the sparse eigenvalue solver EIGS, we find that $\lambda_{\mathrm{pde}} = 0$ is indeed simple as an eigenvalue of $\mathscr{L}$; see Figure 2(a) for the spectrum of $\mathscr{L}$ for values of b in the range $[2,6]$. These results therefore indicate that Hypothesis (H3)(ii) is met, so that there is a family $u_*(x;c)$ of pulses for c near $c_*(b)$ for each fixed $b > 2$.

Next, Hypothesis (H5)(ii) assumes that the function

$$v(x) := \frac{\mathrm{d}}{\mathrm{d}c}u_*(x;c)\Big|_{c=c_*(b)}$$

does not decay faster than exponentially with rate λ^{u} as $x \to \infty$; in other words, $\mathrm{e}^{\lambda^{\mathrm{u}}x}v(x)$ should not converge to zero. Differentiating (17) with respect to c and evaluating at $c = c_*(b)$, we see that the function $v(x)$ satisfies the system

$$\mathscr{L}v + u_*(x) = 0.$$

We calculated this solution in AUTO on the interval $[0,L]$ for with Neumann boundary conditions on either end for different values of L and plotted $\mathrm{e}^{\lambda^{\mathrm{u}}L}v(L)$ for some of the values of L in Figure 2(b). The results indicate that Hypothesis (H5)(ii) is also met.

In summary, a combination of analytical verification and numerical computations indicate that Theorem 1 applies to the fifth-order water-wave problem (17). It remains to evaluate the constants

$$b^j := \langle H_{uu}(0,0)v_j, Rv_j\rangle, \qquad j = \mathrm{u},\mathrm{uu}$$

from (4), where we can take v_{u} and v_{uu} to be any eigenvectors of the linearization

$$JH_{uu}(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -c & 0 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{15}{2} \\ c & 0 & 0 & 0 \\ 0 & b & -1 & 0 \end{pmatrix}$$

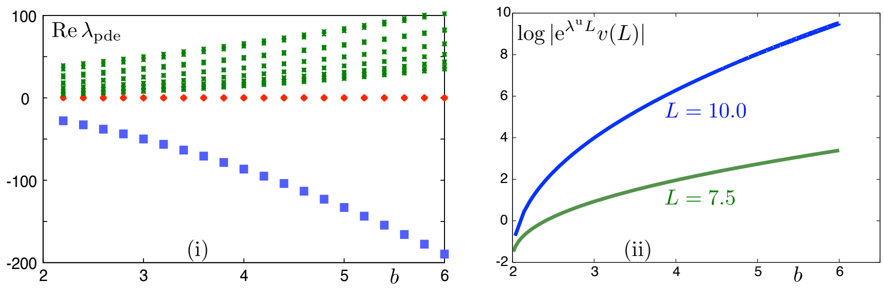

Fig. 2 The numerical computations presented in panel (i) indicate that $\lambda_{\rm pde} = 0$ is simple as an eigenvalue of the linearization $\mathscr{L}$ about the pulse; this indicates that Hypothesis (H3)(ii) is met. In panel (ii), we plot b versus $e^{\lambda^u L}\frac{d}{dc}u_*(x;c)|_{c=c_*(b)}$ for different values of L: this quantity is not decreasing as L increases, thus indicating that Hypothesis (H5)(ii) is also satisfied. We refer to the main text for details on how these computations were carried out.

about $u = 0$ at $c = c_*(b)$. The eigenvector v belonging to a real eigenvalue λ is given by

$$v = \left(1, \lambda, \frac{c}{\lambda}, \frac{2\lambda^2}{15}\right).$$

Thus, upon using (20), we obtain

$$\langle H_{uu}(0)v, Rv\rangle = b\lambda^2 - 2c,$$

and substituting the eigenvalues from (21), we obtain

$$b^u = -\frac{3}{10}(3b+4)(b-2), \qquad b^{uu} = \frac{3}{5}(3b+4)(2b+1).$$

Thus, b^{uu} is positive for $b > -\frac{1}{2}$, while b^u is positive for $-\frac{1}{2} < b < 2$ and negative for $b > 2$. In summary, we expect N-pulses to bifurcate from the primary pulse for $b > 2$, while our theory does not apply to $-\frac{1}{2} < b < 2$ as the origin is not hyperbolic in this parameter range.

The analytical predictions from Theorem 1 (communicated by me to the authors of [2] prior to publication of [2]) were confirmed in the numerical computations of 2-pulses presented in [2, Figures 23-24] below and above the bifurcation point $c = c_*(b)$.

5 Open problems

One of the issues not addressed here, or elsewhere, is the stability of the multi-pulses we found above under the time evolution of the fifth-order model (16)

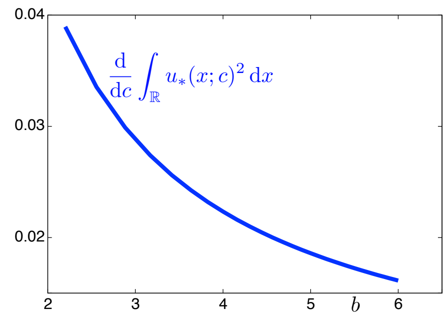

Fig. 3 Shown is the dependence of $\frac{\mathrm{d}}{\mathrm{d}c}\int_{\mathbb{R}} u_*(x;c)^2\,\mathrm{d}x$ on the parameter b. This quantity is computed numerically using AUTO.

$$u_t + \partial_x\left(\frac{2}{15}u_{xxxx} - bu_{xx} + cu + \frac{3}{2}u^2 + \frac{1}{2}u_x^2 + uu_{xx}\right) = 0, \qquad x \in \mathbb{R}, \tag{23}$$

now written in a co-moving frame. This is a difficult question as the PDE (23) is Hamiltonian when posed on appropriate function spaces since it can be written as

$$u_t = -\partial_x E'(u) \tag{24}$$

where $J := -\partial_x$ is skew-symmetric and

$$E(u) = \frac{1}{2}\int_{\mathbb{R}}\left(\frac{2}{15}u_{xx}^2 + bu_x^2 + cu^2 + u^3 - uu_x^2\right)\mathrm{d}x.$$

It is worthwhile to point out that the L^2-norm

$$N(u) = \frac{1}{2}\int_{\mathbb{R}} u(x)^2\,\mathrm{d}x,$$

is invariant under the time evolution of (23). More generally, many other fifth-order equations, such as the fifth-order Korteweg–de Vries equation

$$u_t + \partial_x\left(u_{xxxx} - u_{xx} + cu + u^2\right), \qquad x \in \mathbb{R}, \tag{25}$$

which are of the form (24) with the same conserved quantity $N(u)$, are known to exhibit solitary waves and multi-pulses: it is therefore natural, and indeed important for applications, to investigate the temporal stability of their pulses and multi-pulses.

Before we discuss multi-pulses, we briefly review stability results for the underlying primary solitary wave $u_*(x)$ given in (18) of (23); recall that this profile is in a flip configuration and gives rise to multi-pulses as discussed in the previous section. Given the Hamiltonian nature of (23), it is natural to construct stable stationary solutions of (23), which correspond to traveling waves with speed c of the original equation (1), by seeking minimizers of the Hamiltonian E. However, solitary waves usually do not minimize the functional E. Instead, they can be thought of as constrained minimizers of $\tilde{E}(u) = E(u) - cN(u)$ under the constraint $N(u) = \text{const}$; in this formulation, the wave speed c arises as a Lagrange multiplier. Typically, the

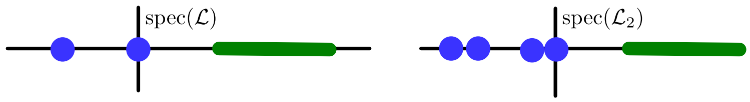

Fig. 4 Shown are the anticipated spectra of the Hessian of the energy $\mathscr{E}$ evaluated at the primary pulse (left) and the 2-pulse (right).

energy will decrease as one moves along the family $u_*(\cdot;c)$ of solitary waves with c varying, while b is kept fixed. As shown in [3, Theorem 2.1] and the references therein, the pulse $u_*(x)$ will be stable for (23) if the Hessian, or second variation, $E''(u_*) = \mathscr{L}$ of E has a simple eigenvalue at the origin and only one negative eigenvalue and if furthermore $\frac{\mathrm{d}}{\mathrm{d}c}N(u_*(\cdot;c)) > 0$. For (23), the first hypothesis is checked numerically in Figure 2(i), while Figure 3 indicates the second assumption is met as well. These numerical calculations therefore indicate that the underlying primary pulses (18) are indeed stable, and it is natural to discuss whether the multi-pulses emerging from them are stable, too.

First, we consider the Hessian $\mathscr{L}_N$ of the PDE energy E at an N-pulse solution. It follows from [11] that $\mathscr{L}_N$ has N eigenvalues near the negative eigenvalue of $\mathscr{L}_1$ and exactly N small eigenvalues near the origin. For (23), preliminary computations that follow [11] indicate that $N-1$ of these small eigenvalues are negative, while the remaining small eigenvalue is at the origin, as dictated by translation invariance of the energy; see Figure 4. In particular, there are now $2N-1$ direction along with the energy decreases, and the single known conserved quantity $N(u)$ cannot compensate for them when $N > 1$.

Next, consider the linearization $J\mathscr{L}_N = -\partial_x \mathscr{L}_N$ of the PDE (24) about an N-pulse. The anticipated spectra of $J\mathscr{L}$ and $J\mathscr{L}_N$ are shown in Figure 5 for $N = 2$. Indeed, the results in [11], applied in an appropriate exponentially weighted norm to the linearization $J\mathscr{L}_N$, show that $J\mathscr{L}$ will have $2N$ eigenvalues near the origin, and two of these will reside at the origin due to translational symmetry. For a 2-pulse, the remaining two eigenvalues may reside on the real or imaginary axis, or move off the imaginary axis. In the latter case, there will be another pair of eigenvalues on the other side of the imaginary axis as the spectrum of Hamiltonian linearizations is symmetric with respect to reflections across the imaginary axis. The reason that the two extra eigenvalues are not included in the count of $2N$ is that their eigenfunctions are not bounded under the exponential weighted norm used to locate them. We claim that the case shown in Figure 5(iii) can actually not occur given that the spectrum of $\mathscr{L}_2$ looks as shown in Figure 4. Indeed, there should be three directions along which the energy decreases near the 2-pulse: one of these directions corresponds again to changing the speed of the 2-pulse, and the other two directions must therefore be associated with the eigenspace of the two real nonzero eigenvalues; however, the dynamics on this eigenspace is of saddle-type, and the energy therefore decreases only along one and not both of these directions. More generally, we expect that an expression of the form

$$n(\mathscr{L}) - 1 = k_\mathrm{r} + 2k_\mathrm{i}^- + 2k_\mathrm{c}$$

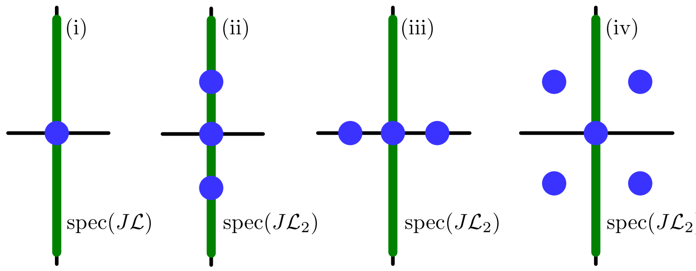

Fig. 5 Panel (i) shows the spectrum of the linearization of (24) about a stable primary pulse. Panels (ii)-(iv) show the anticipated three possibilities for the spectrum of the PDE linearization about a 2-pulse. The origin is an algebraically double eigenvalue in all panels.

holds, where $n(\mathscr{L})$ is the number of strictly negative eigenvalues of $\mathscr{L}$, while k_r, k_i^-, and k_c denote the number of pairs of real eigenvalues, pairs of purely imaginary eigenvalues with negative Krein signature (meaning that $\mathscr{L}$ is negative definite on the associated eigenspace), and quadruplets of genuinely complex eigenvalues, respectively. Theorems of this type can be found, for instance, in [7] and [4], though results of this type are not known for fifth-order KdV equations posed on the real line as the symplectic operator ∂_x is not bounded (and boundedness is a crucial assumption needed in [7] and references therein). To conclude at least spectral stability, it therefore remains to exclude the case shown in Figure 5(iv). This is difficult for the following reason: the computations arising from [11] show that the quadruplet lies, to leading order, on the imaginary axis, and it is not clear how a refined analysis could exclude the possibility of a very small yet nonzero real part for those eigenvalues. Furthermore, even if the eigenvalues start out to be purely imaginary, as shown in Figure 5(ii), then these eigenvalues with negative Krein signature can move off the imaginary axis as soon as they collide with eigenvalues of positive Krein signature[1]. Since the essential spectrum that occupies the imaginary axis has positive Krein signature, there does not seem to be an immediate structural reason that confines these eigenvalues to the imaginary axis. I believe that the eigenvalues lie on the imaginary axis, and one possible way of ascertaining that they do is to use the recent Krein-matrix formalism developed in [6]: this formalism shows that there can be hidden structural reasons that prevent eigenvalues from leaving the imaginary axis even when eigenvalues of opposite Krein signature collide. Currently, the formalism in [6] applies only to problems with discrete spectrum, and it remains an open problem whether it can be extended to PDEs of the form (23).

Finally, I discuss briefly what type of nonlinear stability one might expect if the 2-pulses turn out to be spectrally stable. Following [9], the idea is to work in an appropriate exponentially weighted space in which the spectrum of $J\mathscr{L}_2$ will look as shown in Figure 6. It should then be possible to extend the analysis in [9] to prove the

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[1] The Hessian of the energy restricted to the eigenspace associated with a quadruplet off the imaginary axis must decrease and increase in two transverse planes; thus eigenvalues can leave the imaginary axis only when the energy restricted to their combined eigenspace is indefinite, that is, the eigenvalues have opposite Krein signatures; see [7] and references therein.

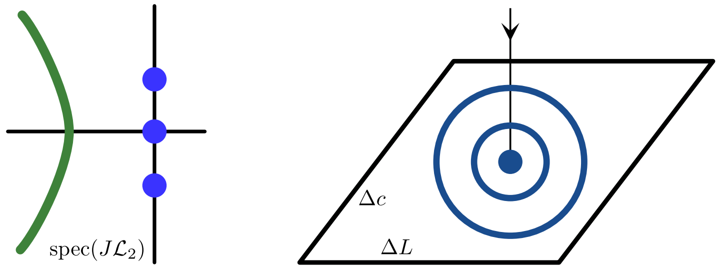

Fig. 6 The left panel indicates the anticipated spectrum of $J\mathscr{L}_2$ in an appropriate exponential weight. The right panel illustrates the anticipated flow on the four-dimensional center manifold, with the two directions that correspond to translation and speed taken out. The energy is still conserved so that the flow must consist of periodic orbits that surround the 2-pulse.

existence of a local four-dimensional center manifold near the 2-pulse the contains the two-parameter family of 2-pulses which is parametrized by their location and speed. The other two directions consist of functions that resemble two copies of the 1-pulse whose distances and relative speeds differ by ΔL and Δc from those of the 2-pulse. This manifold will be exponentially attracting, and nearby solutions will converge to it and asymptotically follow solutions on the center manifold. Since the spectrum of the Hessian is negative definite on the eigenspace associated with the pair of purely imaginary eigenvalues shown in Figure 5(ii), it follows that the flow on the nontrivial part of the center manifold consists of periodic orbits that surround the 2-pulses. Thus, the 2-pulses are expected to be nonlinearly stable in this setting, though, in contrast to the 1-pulse setting of [9], they are not asymptotically stable in the exponential weight.

Most of the discussion above is, of course, highly speculative, though I also believe that some progress on the program outlined above can be made given the recent advances on Krein-signature analyses.

Acknowledgements This paper is dedicated to Jürgen Scheurle on the occasion of his 60th birthday: I am deeply grateful for his expressions of encouragement and support when I began my career as a graduate student and postdoc.

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