Snaking bifurcations of localized patterns on ring lattices

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Abstract

We study the structure of stationary patterns in bistable lattice dynamical systems posed on rings with a symmetric coupling structure in the regime of small coupling strength. We show that sparse coupling (for instance, nearest-neighbour or next-nearest-neighbour coupling) and all-to-all coupling lead to significantly different solution branches. In particular, sparse coupling leads to snaking branches with many saddle-node bifurcations, whilst all-to-all coupling leads to branches with five saddle nodes, regardless of the size of the number of nodes in the graph.

1 Introduction

In this paper, we study the structure of stationary patterns in bistable lattice systems. More precisely, we take a ring of $N \geq 3$ identical nodes and consider the lattice dynamical system given by

$$\dot{u}_n = d(\Delta_m u)_n + f(u_n, \mu), \quad 1 \leq n \leq N \tag{1.1}$$

on this ring for small coupling strength $0 < d \ll 1$. Here, $f(u, \mu)$ is a bistable nonlinearity, and the coupling operator Δ_m denotes the symmetric m-nearest-neighbour connections on the ring; see Figure 1 for an illustration of typical nonlinearities and the coupling structures we consider in this paper. To be specific, the coupling operator in the case $m = 1$ is the usual discrete diffusive (nearest-neighbour) operator

$$(\Delta_1 u)_n = u_{n+1} + u_{n-1} - 2u_n,$$

the case $m = 2$ results in the next-nearest-neighbour coupling

$$(\Delta_2 u)_n = u_{n+2} + u_{n+1} + u_{n-1} + u_{n-2} - 4u_n,$$

and the general case is given by

$$(\Delta_m u)_n = \begin{cases} -(2m+1)u_n + \sum_{j=-m}^{m} u_{n+j}, & 1 \leq m < \lfloor \frac{N}{2} \rfloor \\ -Nu_n + \sum_{j=1}^{N} u_j, & m = \lfloor \frac{N}{2} \rfloor, \end{cases}$$

where we will always take all indices in the set $\{1, \ldots, N\}$ modulo N so that, for instance, $u_{N+1} = u_1$ and $u_{-1} = u_N$. We refer to the case $m = \lfloor \frac{N}{2} \rfloor$ as all-to-all coupling as every element on the ring is connected to every other element.

Our goal is to understand how the arrangement of stationary patterns in the (u, μ) configuration space depends on the interaction length m of the coupling on the ring. Steady states of (1.1) correspond to solutions to the system

$$\mathcal{F}(u, \mu, d) = 0 \tag{1.2}$$

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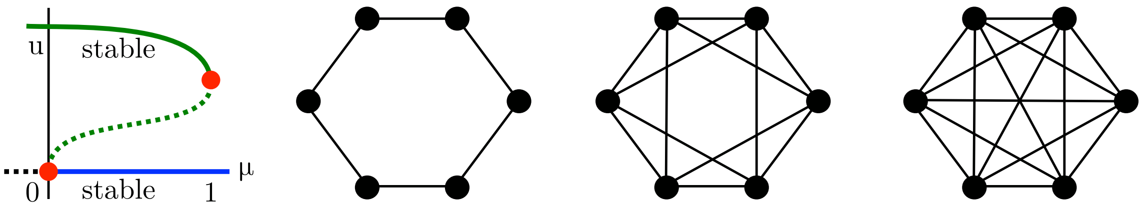

Figure 1: *The left panel illustrates the zero set f(u;) : f(u;) = 0g of the typical bistable nonlinearities f(u;) we will consider. The three rightmost panels contain graphs that consist of N = 6 identical nodes with (from left to right) nearest neighbour, next-nearest neighbour, and all-to-all coupling, respectively.*

where

$$\mathcal{F} : \mathsf{R}^N \times \mathsf{R} \times \mathsf{R} \longrightarrow \mathsf{R}^N, \quad (u, \mu, d) \longmapsto \mathcal{F}(u, \mu, d), \quad \mathcal{F}(u, \mu, d)_n = d(\Delta_m u)_n + f(u_n, \mu).$$

We focus on the case $0 < d \ll 1$, which allows us to exploit that the system (1.2) is uncoupled when $d = 0$. Setting $d = 0$ and choosing $\mu \in (0, 1)$, we select the solution of (1.2) for which u_1 lies on the upper stable branch of the graph of $f(u, \mu)$ and the remaining nodes $u_j = 0$ vanish for $j \neq 1$. We are then interested in describing the connected component of the solution set $\{(u, \mu, d) : \mathcal{F}(u, \mu, d) = 0\}$ that this solution belongs to and understand how this connected component changes as the interaction range m varies.

This question has attracted much attention for coupled lattice systems posed on the 1D and 2D integer lattices. In these spatially extended systems, snaking occurs: for each fixed value of $0 < d \ll 1$, the solution branch consists of an unbounded curve that oscillates back and forth between saddle nodes that occur near $\mu = 0$ and $\mu = 1$ while the norm of the u-component increases without bound as illustrated in the left panel of Figure 2. More generally, this complex bifurcation structure of localized solutions has been observed for a range of spatially discrete systems on integer lattices in [4, 5, 6, 7, 8, 9, 10, 11, 12] and was explained in part by [1, 2, 3]. While the latter works have explained the bifurcation structure of localized solutions on 'regular' graphs, such as the integer lattices, little is known about how graph structure and connection topology influence the connections of localized solutions in parameter space.

In this paper, we focus on snaking in finite rings. In particular, we will provide analytical results for two distinct coupling regimes, namely sparse coupling (with nearest-neighbour and next-nearest-neighbour coupling) and all-to-all coupling. We will prove that the resulting bifurcation curves differ significantly as illustrated in Figure 2: sparse coupling leads to snaking curves with many saddle nodes (depending on the number N of nodes), while all-to-all coupling will always result in a closed curve with five saddle nodes that terminates at a homogeneous state. We also provide illustrative examples in the case of almost all-to-all coupling where additional symmetries in the system result in entirely unique bifurcation curves that would be difficult to predict. We conjecture that sparse coupling will lead to snaking curves that exhibit $\lfloor \frac{N}{2} \rfloor$ saddle nodes for each m with $1 \leq m < \lfloor \frac{N}{2} \rfloor - 1$ (thus excluding almost all-to-all coupling).

2 Main results

Recall that we are interested in the solution structure of the system

$$\dot{u}_n = d(\Delta_m u)_n + f(u_n, \mu), \quad 1 \leq n \leq N. \tag{2.1}$$

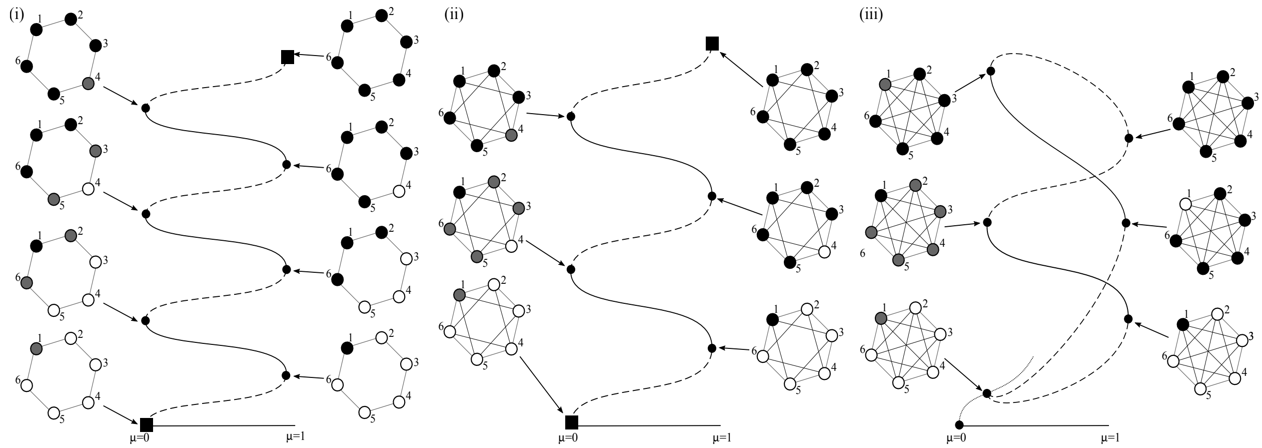

Figure 2: *Di erent bifurcation curves depending on the number of symmetric connections over a six element ring. (i) Nearest-neighbour connections lead to a typical snaking bifurcation diagram, as is proven in Theorem 1, (ii) Next-nearest-neighbour connections on a six element ring introduce a further symmetry for which the elements indexed by 2,3,5,6 bifurcation together near* $= 0$*, as proven in Theorem 2. (iii) In the case of all-to-all coupling, the system is invariant with respect to any permutation of the nodes on the ring and so the bifurcation curves form a small closed curve that bifurcates from the homogeneous state (dotted curve) near the origin, as proven in Theorem 4. Squares marking the end of the curves in panels (i) and (ii) represent the set of exceptional bifurcations which are not proven in this work.*

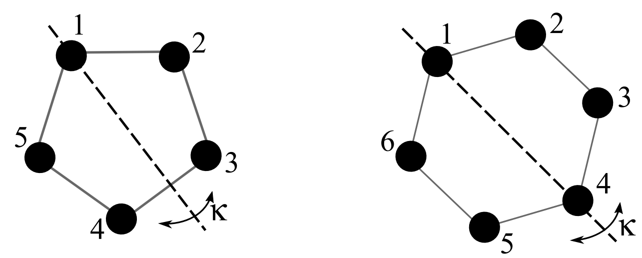

Figure 3: *Action of the ip on odd and even element rings. On the left is a ring of* $N = 5$ *elements, for which the ip leaves only the element at index* 1 *xed, while the right presents a ring of* $N = 6$ *elements, for which the ip leaves the elements at index* 1 *and* 4 *xed.*

First, note that system (2.1) is equivariant with respect to the dihedral symmetry group D_N, generated by the actions

$$\begin{aligned} \zeta(u_1, u_2, \dots, u_N) &= (u_2, \dots, u_N, u_1) \\ \kappa(u_1, u_2, u_3, \dots, u_{N-1}, u_N) &= (u_1, u_N, u_{N-1}, \dots, u_3, u_2) \end{aligned} \tag{2.2}$$

which define rotations and flips of the ring. Clearly $\zeta^N = 1$ and $\kappa^2 = 1$, where we use 1 to denote the identity element that acts trivially on vectors in $\mathbb{R}^N$. Our interest in what follows will lie in solutions of (2.1) that are invariant with respect to the action of κ. Notice that the action of κ on rings with N even leaves exactly two elements fixed, while when N is odd only the element at index 1 is fixed. We refer the reader to Figure 3 for a demonstration of these two cases with $N = 5, 6$. Finally, in the case of all-to-all coupling, system (2.1) is equivariant with respect to the symmetric group of permutations on N elements, which is a strictly larger class of symmetries than those for all other classes of coupling functions considered in this work.

Here the function f is assumed to be bistable, satisfying the same properties as in [3]. We summarize these properties in the following hypothesis; see also Figure 1.

Hypothesis 1. *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is smooth and satis es the following:*

(i) The function f is odd in u so that $f(-u,\mu) = -f(u,\mu)$ for all (u,μ).

(ii) The set of roots of $f(u,\mu)$ is as shown in the left panel of Figure 1. In particular, for each $\mu \in (0,1)$, the function $f(u,\mu)$ has exactly three nonnegative zeros, namely $u = 0$ and $u = u_\pm(\mu)$ with $0 < u_-(\mu) < u_+(\mu)$, and these satisfy $f_u(0,\mu), f_u(u_+(\mu),\mu) < 0 < f_u(u_-(\mu),\mu)$.

(iii) At $\mu = 0$, the zeros $u = 0$ and $u = \pm u_-(\mu)$ collide in a generic subcritical pitchfork bifurcation.

(iv) At $\mu = 1$, the zeros $u = u_\pm(\mu)$ collide in a generic saddle-node bifurcation.

We note that all of our analysis can be applied to bistable functions for which the bifurcation at $\mu = 0$ or $\mu = 1$ is instead a transcritical bifurcation, as was shown to be true in [3]. To precisely state our results we instead focus on functions satisfying Hypothesis 1 with the prototypical example being the cubic-quintic nonlinearity

$$f(u,\mu) = -\mu u + 2u^3 - u^5. \tag{2.3}$$

Throughout this section we detail our results on localized pattern formation in the lattice system (2.1) with functions f satisfying Hypothesis 1 based on the value of m. All proofs will be left to Section 3.

2.1 Sparse coupling

We begin with the case of 'sparse' coupling, i.e. $m \ll \lfloor N/2 \rfloor$. In particular, we will focus on the cases of $m = 1, 2$ to show that localized solutions grow around the ring in the form of a snaking bifurcation curve with the region of activation growing symmetrically around the ring as one ascends the diagram, as is illustrated in panel (i) of Figure 2. Since we are interested in solutions which are invariant with respect to the action of κ, we can restrict our attention to the index set

$$I = \left\{1 \leq n \leq \left\lfloor \frac{N}{2} \right\rfloor + 1\right\} \tag{2.4}$$

for any fixed $N \geq 2$. Indeed, an element $u \in \mathbb{R}^N$ satisfying $\kappa u = u$ is uniquely identified by elements with indices belonging to I. The reader is referred to Figure 3, where we see that when $N = 5$ we have $I = \{1, 2, 3\}$ and the elements at indices $4, 5$ are identical to those at $3, 2$, respectively. Similarly, from Figure 3, when $N = 6$ we have $I = \{1, 2, 3, 4\}$ and the elements at indices $5, 6$ are identical to those at $3, 2$, respectively.

Hypothesis 1 implies that any nonnegative solution of (3.2) with $d = 0$ must have $u_n \in \{0, u_\pm(\mu)\}$ when $\mu \in [0,1]$. Then, for each $0 \leq \mu \leq 1$ and $k \in I$ we define the elements $\bar{u}^{(k)}(\mu) = \{\bar{u}_n^{(k)}(\mu)\}_{n\in I}$ and $\bar{v}^{(k)}(\mu) = \{\bar{v}_n^{(k)}(\mu)\}_{n\in I}$ by

$$\bar{u}_n^{(k)}(\mu) = \begin{cases} u_+(\mu) & 1 \leq n \leq k \\ 0 & n > k \end{cases} \tag{2.5}$$

and

$$\bar{v}_n^{(k)}(\mu) = \begin{cases} u_+(\mu) & 1 \leq n < k \\ u_-(\mu) & n = k \\ 0 & n > k \end{cases} \tag{2.6}$$

for all $n \in I$ and $\mu \in [0,1]$. Notice that when $k = \lfloor N/2 \rfloor + 1$ we have that $\bar{u}^{(k)}$ is a uniform state with all entries given by $u_+(\mu)$. From the discussion above we have that these elements can be extended to κ-invariant solutions of (1.2) by setting u_n for $n \notin I$ to be defined by the relation $\kappa u = u$.

The elements (2.5) and (2.6) are pairwise distinct when $\mu \in (0,1)$, but Hypothesis 1 gives that $\lim_{\mu\to 0^+} u_-(\mu) =$

0 and $\lim_{\mu\to 1} \; u_-(\mu) = u_+(1)$, and so that the patterns (2.5) and (2.6) satisfy

$$\begin{aligned} \bar{u}^{(k-1)}(0) &= \bar{v}^{(k)}(0), \quad k = 2, \dots, \left\lfloor \frac{N}{2} \right\rfloor + 1 \\ \bar{u}^{(k)}(1) &= \bar{v}^{(k)}(1), \quad k = 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor \end{aligned} \tag{2.7}$$

at the parameter boundaries $\mu = 0, 1$. We can therefore define the connected set

$$\Gamma_{\text{sparse}} := \bigcup_{k\in I} \bigcup_{0\leq\mu\leq 1} \{(\bar{u}^{(k)}(\mu), \mu), (\bar{v}^{(k)}(\mu), \mu)\} \subset \mathsf{R}^N \times [0, 1], \tag{2.8}$$

which represents the union of the curves traced out for $\mu \in [0, 1]$ by the patterns (2.5) and (2.6) of (1.2) when $d = 0$. We will also define the exceptional set $\mathcal{E}_{\text{sparse}}$, given by

$$\mathcal{E} := \{\bar{v}^{(1)}(0)\} \cup \{\bar{v}^{(\lfloor \frac{N}{2} \rfloor + 1)}(1)\}, \tag{2.9}$$

representing the endpoints of the curve Γ_{sparse}. We present the following theorem, for which Figure 2(i) provides an illustration of the results for $(N, m) = (6, 1)$. The proofs for $m = 1$ are left to §3.1 and $m = 2$ is left to §3.2.

Theorem 1. *Assume that f satis es Hypothesis 1. If $N \geq 4$ and $m = 1$ or $N \geq 7$ and $m = 2$, then for each $\delta_* > 0$ there exists $d_* > 0$ such that the set $U_\delta\ (\Gamma_{\text{sparse}}) \setminus U_{2\delta}\ (\mathcal{E})$ contains a unique, nonempty, continuous branch of κ-symmetric solutions of the steady-state system (1.2). Furthermore, this branch is smooth and C^1-close to Γ_{sparse} for each d, depends smoothly on d, and its limit as $d \to 0^+$ is contained in Γ_{sparse}.*

2.2 Almost all-to-all coupling

Notice that in the statement of Theorem 1 we were required to take $N \geq 7$ when $m = 2$ to obtain the snaking bifurcation curves of localized solutions similar to those with $m = 1$. This is because when N is even and $m = \lfloor \frac{N}{2} \rfloor - 1$, new symmetries are introduced into the model. This has the effect that the resulting bifurcation diagram is markedly different than the usual snaking diagram for sparse coupling, while also being distinct from the fully symmetric case of all-to-all coupling covered in the following subsection. It appears that many of these bifurcations curves must be understood on a case-by-case basis for varying N, and so instead of attempting to exhaustively document all of these cases, we opt to illustrate with two specific examples. The cases detailed here will take $N = 6, 8$, representing the smallest ring sizes where these atypical bifurcation diagrams can be observed.

Let us begin with $N = 6$. Much of the bifurcation structure in the case when $(N, m) = (6, 2)$ is similar to that of the case when $N \geq 7$ and $m = 2$, with the following exception: the connection from the continued solutions of $\bar{u}^{(1)}(\mu)$ to $\bar{v}^{(2)}(\mu)$, defined in (2.5) and (2.6), respectively, near $(d, \mu) = (0, 0)$ is not present. This connection is replaced by a connection from the continued solutions of $\bar{u}^{(1)}(\mu)$ to the branch continued from the solution $\bar{w}^{(23)}(\mu)$, given by

$$\bar{w}_n^{(23)} = \begin{cases} u_+(\mu) & n = 1 \\ u_-(\mu) & n = 2, 3 \\ 0 & n = 4 \end{cases} \tag{2.10}$$

for each $\mu \in [0, 1]$. Since the solution $\bar{w}^{(23)}(\mu)$ satisfies $\lim_{\mu\to 0^+} \bar{w}^{(23)}(\mu) = \bar{u}^{(1)}(0)$ and $\lim_{\mu\to 1} \; \bar{w}^{(23)}(\mu) = \bar{u}^{(3)}(1)$, we may define the connected set

$$\Gamma_{6,2} := \bigcup_{0\leq\mu\leq 1} \{(\bar{v}^{(1)}, \mu), (\bar{u}^{(1)}, \mu), (\bar{w}^{(23)}, \mu), (\bar{u}^{(3)}, \mu), (\bar{v}^{(4)}, \mu)\} \subset \mathsf{R}^6 \times [0, 1]. \tag{2.11}$$

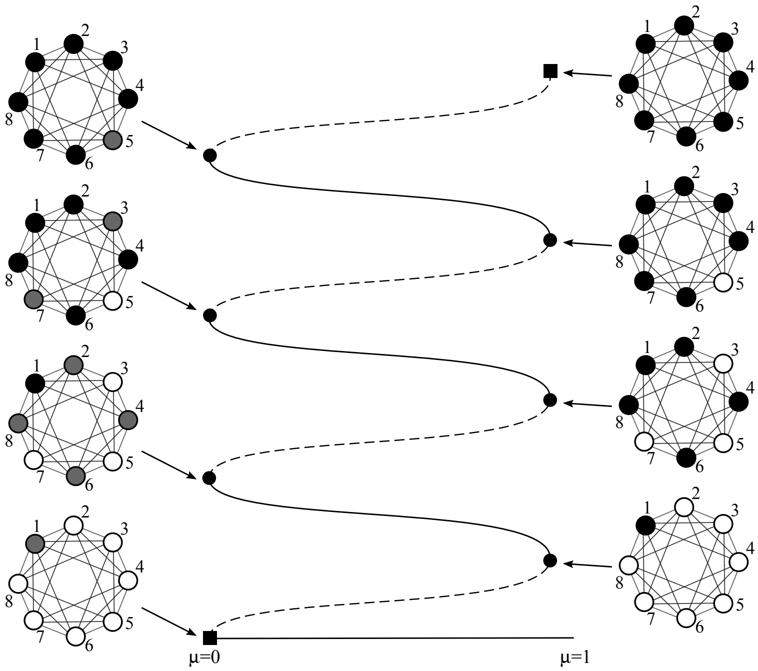

Figure 4: *The bifurcation diagram on a ring with* $(N; m) = (8; 3)$ *for small* $d > 0$. *The number of neighbour connections induces an added symmetry, thus leading to the above bifurcation diagram which bears little resemblance to that of the case* $(N; m) = (8; 1); (8; 2)$ *featured in Theorem 1. Squares marking the end of the curve represent the set of exceptional bifurcations, continued from E, which are not proven in this work.*

Using again $\mathcal{E}$ defined in (2.9), this leads to the following theorem whose results are presented visually in Figure 2(ii). The proof is left to §3.3.

Theorem 2. *Assume that* f *satis es Hypothesis 1 and that* $(N, m) = (6, 2)$. *Then for each* $\delta_* > 0$ *there exists* $d_* > 0$ *such that for each* $0 < d < d_*$ *the set* $U_\delta(\Gamma_{6,2}) \setminus U_{2\delta}(\mathcal{E})$ *contains a nonempty, continuous branch of* κ*-symmetric solutions of the steady-state system* (1.2). *Furthermore, this branch is smooth and* C^1*-close to* $\Gamma_{6,2}$ *for each* d*, depends smoothly on* d*, and its limit as* $d \to 0^+$ *is contained in* $\Gamma_{6,2}$.

Turning now to the case $(N, m) = (8, 3)$, we are required to define two more patterns which correspond to κ-invariant solutions of (3.2) with $d = 0$. Consider the elements $\bar{w}_{\pm}^{(24)}(\mu)$ and $w_{-}^{(3)}(\mu)$, given by

$$\bar{w}_{\pm,n}^{(24)}(\mu) = \begin{cases} u_+(\mu) & n = 1 \\ u_\pm(\mu) & n = 2, 4 \\ 0 & n = 3, 5 \end{cases} \tag{2.12}$$

and

$$\bar{w}_{-,n}^{(3)}(\mu) = \begin{cases} u_+(\mu) & n = 1, 2, 4 \\ u_-(\mu) & n = 3 \\ 0 & n = 5 \end{cases} \tag{2.13}$$

respectively, for each $\mu \in [0, 1]$. Here the values in the superscript detail which terms are at $u_-(\mu)$ when the subscript is $-$, while the element $\bar{w}_{+}^{(24)}(\mu)$ connects the two elements with $-$ subscripts. We refer the reader to panels (ii) and (iii) of Figure 5 below for visual depictions of these elements extended by κ symmetry to the entire ring. Notice that we have the following connections:

$$\begin{aligned}\lim_{\mu\to 0^+} \bar{w}_-^{(24)}(\mu) &= \lim_{\mu\to 0^+} \bar{u}^{(1)}(\mu)\\ \lim_{\mu\to 1} \bar{w}_+^{(24)}(\mu) &= \lim_{\mu\to 1} \bar{w}_-^{(3)}(\mu)\\ \lim_{\mu\to 0^+} \bar{w}_-^{(3)}(\mu) &= \lim_{\mu\to 0^+} \bar{u}^{(4)}(\mu).\end{aligned} \tag{2.14}$$

These connections allow one to define the connected set

$$\Gamma_{8,3} := \bigcup_{0\le\mu\le 1} \{(\bar{v}^{(1)},\mu),(\bar{u}^{(1)},\mu),(\bar{w}_-^{(24)},\mu),(\bar{w}_+^{(24)},\mu),(\bar{w}_-^{(3)},\mu),(\bar{u}^{(4)},\mu),(\bar{v}^{(4)},\mu)\} \subset \mathsf{R}^8\times[0,1]. \tag{2.15}$$

This leads to the following result whose results are represented visually in Figure 4. The proof is again left to §3.3.

Theorem 3. *Assume that f satis es Hypothesis 1 and that $(N,m) = (8,3)$. Then for each $\delta_* > 0$ there exists $d_* > 0$ such that for each $0 < d < d_*$ the set $U_\delta\ (\Gamma_{8,3}) \setminus U_{2\delta}\ (\mathcal{E})$ contains a nonempty, continuous branch of κ-symmetric solutions of the steady-state system (1.2). Furthermore, this branch is smooth and C^1-close to $\Gamma_{8,3}$ for each d, depends smoothly on d, and its limit as $d \to 0^+$ is contained in $\Gamma_{8,3}$.*

2.3 All-to-all coupling

Let us now consider the case of all-to-all coupling, i.e. $m = \lfloor \frac{N}{2} \rfloor$. In this case (1.2) becomes

$$d\sum_{j=1}^{N}(u_j - u_n) + f(u_n,\mu) = 0, \tag{2.16}$$

so that every element is coupled to every other element. Alternatively, the coupling topology can be represented by a complete graph with N vertices. Importantly, the symmetry group associated to (2.16) is much larger than just the dihedral group D_N and is given by the group of all permutations of the vectors $(u_1, u_2, \dots, u_N) \in \mathsf{R}^N$. This increased symmetry in the lattice system means that we can extend our search to localized solutions which are not necessarily κ-symmetric with almost no difference to the analysis. Instead we will consider for each $k = 1, \dots, N-1$ the symmetry group G_k defined by the set of all permutations of the first k elements of a vector in R^N together with all permutations of the last $N-k$ elements of the vector. We note that the fixed point space of G_k in R^N is the set of all vectors whose first k elements are identical and whose last $N-k$ elements are identical. That is, the fixed point space is two-dimensional.

Proceeding as in the previous subsections, for each $k = 1, \dots, N-1$ let us define the G_k-invariant vectors $\bar{a}_\pm^{(k)}(\mu) = \{\bar{a}_{\pm,n}^{(k)}(\mu)\}$, $\bar{b}^{(k)}(\mu) = \{\bar{b}_n^{(k)}(\mu)\}$, $\bar{c}_\pm^{(k)}(\mu) = \{\bar{c}_{\pm,n}^{(k)}(\mu)\}$, and $\bar{d}^{(k)}(\mu) = \{\bar{d}_n^{(k)}(\mu)\}$, given by

$$\bar{a}_{\pm,n}^{(k)}(\mu) = \begin{cases} u_\pm(\mu) & 1 \le n \le k \\ 0 & n > k \end{cases} \tag{2.17}$$

$$\bar{b}_n^{(k)}(\mu) = \begin{cases} u_+(\mu) & 1 \le n \le k \\ u_-(\mu) & n > k \end{cases} \tag{2.18}$$

$$\bar{c}_{\pm,n}^{(k)}(\mu) = \begin{cases} 0 & 1 \le n \le k \\ u_\pm(\mu) & n > k \end{cases} \tag{2.19}$$

and

$$\bar{d}_n^{(k)}(\mu) = \begin{cases} u_-(\mu) & 1 \le n \le k \\ u_+(\mu) & n > k \end{cases} \tag{2.20}$$

for each $\mu \in [0,1]$. From Hypothesis 1 we have that for any k the vectors (2.17) – (2.20) are G_k-invariant solutions of (1.2) when $d = 0$. Furthermore, we have the following connections:

$$\begin{aligned} \lim_{\mu\to 1} \bar{a}_-^{(k)}(\mu) &= \lim_{\mu\to 1} \bar{a}_+^{(k)}(\mu) \\ \lim_{\mu\to 0^+} \bar{a}_+^{(k)}(\mu) &= \lim_{\mu\to 0^+} \bar{b}^{(k)}(\mu) \\ \lim_{\mu\to 1} \bar{b}^{(k)}(\mu) &= \lim_{\mu\to 1} \bar{d}^{(k)}(\mu) \\ \lim_{\mu\to 0^+} \bar{c}_+^{(k)}(\mu) &= \lim_{\mu\to 0^+} \bar{d}^{(k)}(\mu) \\ \lim_{\mu\to 1} \bar{c}_-^{(k)}(\mu) &= \lim_{\mu\to 1} \bar{c}_+^{(k)}(\mu) \end{aligned} \tag{2.21}$$

for each k. We therefore introduce the connected curve

$$\Gamma_{\text{all}}^k := \bigcup_{0\le\mu\le 1} \{(\bar{a}_\pm^{(k)}(\mu),\mu), (\bar{b}^{(k)}(\mu),\mu), (\bar{c}_\pm^{(k)}(\mu),\mu), (\bar{d}^{(k)}(\mu),\mu)\} \subset \mathsf{R}^N \times [0,1], \tag{2.22}$$

for each $k = 1,\dots,N-1$. We present the following result, whose proof is left to §3.4.

Theorem 4. *Assume that f satis es Hypothesis 1. Then for each $k = 1,\dots,N-1$ and $\delta_* > 0$, there exists $d_* > 0$ such that for each $0 < d < d_*$ the set $U_\delta\ (\Gamma_{\text{all}}^k)\backslash U_{2\delta}\ (\{(0,0)\})$ contains a unique, nonempty, continuous branch of G_k-symmetric solutions of the steady-state system (2.16). In the neighbourhood $U_{2\delta}\ (\{(0,0)\})$ this branch terminates onto the homogeneous branch of solutions to (1.2) given by $u_n = u_-(\mu)$ for all $n = 1,\dots,N$ at $\mu = \frac{N}{2}d + \mathcal{O}(d^2)$. Furthermore, this branch is smooth and C^1-close to Γ_{all}^k for each d, depends smoothly on d, and its limit as $d \to 0^+$ is contained in Γ_{all}^k.*

We note that the results of Theorem 4 do not include an exceptional set, meaning that the theorem completely characterizes the entire bifurcation curve of localized solutions in the presence of all-to-all coupling. We refer to Figure 2(iii) for a visual depiction of the result with a ring of $N = 6$ elements. Interestingly, we see that this curve of solutions does not originate at the origin $(u,\mu) = (0,0)$ in $\mathsf{R}^N \times [0,1]$ when $0 < d \ll 1$, but instead bifurcates from the homogenous branch of solutions at a positive value of μ. For more details on this bifurcation from the homogenous branch, we refer the reader to Lemma 3.15 below.

3 Proofs

In this section we will prove the various theorems from the previous section detailing the existence and bifurcation structure of steady-state solutions to (2.1). As discussed in the introduction, we can define the function

$$\mathcal{F}:\ \mathsf{R}^N \times \mathsf{R} \times \mathsf{R} \longrightarrow \mathsf{R}^N, \quad (u,\mu,d) \longmapsto \mathcal{F}(u,\mu,d), \quad \mathcal{F}(u,\mu,d)_n := d(\Delta_m u)_n + f(u_n,\mu), \tag{3.1}$$

to see that the steady-state system

$$d(\Delta_m u)_n + f(u_n,\mu) = 0 \tag{3.2}$$

corresponding to (2.1) is given by $\mathcal{F}(u,\mu,d) = 0$. Note that $\mathcal{F}$ is smooth in its arguments, and upon taking $d = 0$, solving $\mathcal{F}(u,\mu,0) = 0$ with $\mu \in (0,1)$ reduces to taking $u_n \in \{0, \pm u_\pm(\mu)\}$ for each $n = 1,\dots N$ per

Hypotheses 1. From the assumption on the non-degeneracy of the roots of f, it follows that for μ belonging to any compact interval of $(0,1)$ these solutions may be continued regularly into $d>0$ using the implicit function theorem. For more details on the proof of these facts, the reader is referred to [3, Lemma 3.1].

Throughout these proofs we will make use of the following. Using a change of coordinates, we can bring the Taylor expansion of $f(u,\mu)$ about $(0,0)$ into the form

$$f(u,\mu) = -\mu u + u^3 + \mathcal{O}(\mu^2 u + \mu u^3 + u^5) \tag{3.3}$$

and we may also assume that $u_+(0) = 1$. Similarly, upon changing coordinates we can bring the Taylor expansion of $f(u,\mu)$ about $(1,1)$ into the form

$$f(1+u, 1+\mu) = -\mu - u^2 + b\mu u + \mathcal{O}(\mu^2 + \mu u^2 + u^4). \tag{3.4}$$

These changes of coordinates will greatly simplify the analysis in what follows.

3.1 Nearest-neighbour coupling

Throughout this section we will consider $N \geq 4$ and take $m = 1$, representing nearest-neighbour connections in the system (1.1). In sum, this subsection proves Theorem 1 with $m = 1$, while the case of $m = 2$ is left to the following subsection. Per the discussion at the beginning of this section, we need only continue the connections connections in $\Gamma_{\text{sparse}} \setminus \mathcal{E}_{\text{sparse}}$ near $\mu = 0, 1$. We begin with the following lemma which continues the patterns near $\mu = 0$.

Lemma 3.1. *Fix $m = 1$ and $2 \leq k \leq \lfloor \frac{N}{2} \rfloor + 1$, then the following is true for (3.2). There are constants $d_1, \mu_1 > 0$ and a smooth function $\mu_l : [0, d_1] \to [0, \mu_1]$ such that for each $d \in (0, d_1]$ there is a pair of κ-symmetric solutions $u_l(\mu, d)$ and $v_l(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_l(d)$ and exist for all $\mu \in [\mu_l(d), \mu_1]$. These solutions are smooth in (μ, d), and for each xed μ, we have $u_l(\mu, d) \to \bar{u}^{(k-1)}(\mu)$ and $v_l(\mu, d) \to \bar{v}^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_l(d)$ satis es $\mu_l(d) = \frac{3}{\sqrt[3]{2}} d^{\frac{2}{3}} + \mathcal{O}(d)$.*

Proof. We will fix $k \in \{2, \ldots, \lfloor \frac{N}{2} \rfloor + 1\}$ and construct symmetric solutions of (3.2) near the pattern

$$\bar{u}^{(k)}(0) = \bar{v}^{(k)}(0) = \begin{cases} 1 & n < k \\ 0 & \text{otherwise} \end{cases}$$

for (μ, d) near zero. We reduce patterns to the index set I, defined in (2.4), using the aforementioned κ symmetry. To solve $\mathcal{F}(u, \mu, d) = 0$, we note that $\mathcal{F}(\bar{u}^{(k)}(0), 0, 0) = 0$ and that the linearization of $\mathcal{F}$ is given by

$$(\mathcal{F}_u(\bar{u}^{(k)}(0), 0, 0)v)_n = \begin{cases} f_u(1,0)v_n & n < k \\ 0 & n \geq k. \end{cases}$$

Writing $u^+ := u|_{n<k}$ and $u^c := u|_{n \geq k}$, and using that $f_u(1,0) \neq 0$, we can apply the implicit function theorem to conclude that $\mathcal{F}(u, \mu, d) = 0$ restricted to the index set $1 \leq n < k$ has a unique solution $u^+(u^c, \mu, d) \in \mathsf{R}^{k-1}$ for each $u^c \in \mathsf{R}^{\lfloor \frac{N}{2} \rfloor - k + 1}$ and (μ, d) near zero. Furthermore, this solution depends smoothly on its arguments, and so we have

$$u^+(u^c, \mu, d) = 1 + \mathcal{O}(|\mu| + |d| \|u^c\|). \tag{3.5}$$

To solve (3.2) for the indices $n \geq k$, we introduce the scaling

$$\mu = \nu^2, \qquad d = \nu^3 \tilde{d}, \qquad u_n = \nu^{n-k+1} \tilde{u}_n, \tag{3.6}$$

for $n \geq k$ with $|\nu| \ll 1$. Substituting these expressions into (3.2) for each $n \geq k$, we see that (3.2) restricted to the index set $I \setminus I_+$ becomes

$$\begin{aligned} n = k: &\qquad 0 = \nu^3(\tilde{d} - \tilde{u}_k + \tilde{u}_k^3) + \mathcal{O}(\nu^4) \\ n > k: &\qquad 0 = \nu^{n-k+3}(\tilde{d}\tilde{u}_{n-1} - \tilde{u}_n) + \mathcal{O}(\nu^{n-k+4}), \end{aligned}$$

where we recall that we have introduced a change of variable to bring the system to the normal form (3.3). Upon dividing by the leading factors in ν, we arrive at the system

$$\begin{aligned} n = k: &\qquad 0 = \tilde{d} - \tilde{u}_k + \tilde{u}_k^3 + \mathcal{O}(\nu) \\ n > k: &\qquad 0 = \tilde{d}\tilde{u}_{n-1} - \tilde{u}_n + \mathcal{O}(\nu) \end{aligned} \tag{3.7}$$

for which we can see that at the index $n = k$ a fold bifurcation takes place at $(\tilde{u}_k, \tilde{d}, \nu) = (\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}, 0)$. Extending this fold bifurcation to the full system for the indices $n \geq k$ and into $\nu > 0$ now follows as in [3, Lemma 3.2], and is therefore omitted. This completes the proof. □

We now turn to the continuation when $\mu = 1$. Recall that from (3.4) we have that upon changing coordinates we can bring the Taylor expansion of $f(u, \mu)$ about $(1, 1)$ into the form

$$f(1 + u, 1 + \mu) = -\mu - u^2 + \mathcal{O}(\mu^2 + \mu u^2 + u^4).$$

This leads to the following result which continues the connections in Γ near $\mu = 1$ and completes the proof of Theorem 1 when $m = 1$.

Lemma 3.2. *Fix $m = 1$ and $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$, then the following is true for (3.2). There exist constants $d_2, \mu_2 > 0$ and a smooth function $\mu_r : [0, d_2] \to [\mu_2, 1]$ such that for each xed $d \in (0, d_2]$ there is a pair of κ-symmetric solutions $u_r(\mu, d)$ and $v_r(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_r(d)$ and exist for all $\mu \in [\mu_2, \mu_r(d)]$. These solutions are smooth in (μ, d), and for each xed μ we have $u_r(\mu, d) \to \bar{u}^{(k)}(\mu)$ and $v_r(\mu, d) \to \bar{v}^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_r(d)$ is given by $\mu_r(d) = 1 - d + \mathcal{O}(d^{\frac{3}{2}})$.*

Proof. We again restrict to the index set I and extend the solutions by symmetry to $n \geq \lfloor N/2 \rfloor + 2$. As the branch passes near $\mu = 1$, the cell u_k changes from $u_-(\mu)$ to $u_+(\mu)$, while the remaining cells stay near 0 or $u_+(\mu)$. We have that $\mathcal{F}(\bar{u}^{(k)}(1), 1, 0) = 0$ and that the linearization of $\mathcal{F}$ about this solution is given by

$$(\mathcal{F}_u(\bar{u}^{(k)}(1), 1, 0)v)_n = \begin{cases} f_u(0, 1)v_n & n > k \\ 0 & n \leq k. \end{cases}$$

Writing $u^0 := u|_{n>k}$ and $u^c := u|_{n \leq k}$, and using that $f_u(0, 1) \neq 0$, we can apply the implicit function theorem to find that $\mathcal{F}(u, \mu, d) = 0$ restricted to the index set $n > k$ has a unique solution $u^0(u^c, \mu, d) \in \ell^\infty(I|_{n>k})$ for each $u^c \in \mathsf{R}^k$ and (μ, d) near $(1, 0)$. This solution depends smoothly on its arguments, and in particular, has the expansion

$$u^0(u^c, \mu, d) = \mathcal{O}(|\mu - 1| + |d|\|u^c\|). \tag{3.8}$$

To solve (3.2) on the index set $n \leq k$, we introduce the scaling

$$\mu = 1 - \nu^2, \quad d = \nu^2 \tilde{d}, \quad u_{n,m} = 1 + \nu \tilde{u}_n,$$

where $1 \leq n \leq k$ and $|\nu| \ll 1$. Expanding $\mathcal{F}(u, \mu, d) = 0$ restricted to the index set $n \in \{1, \dots, k\}$ in powers of ν and dividing by the leading factor in ν, we arrive at the finite system

$$\begin{aligned} n = k: &\qquad 0 = -\tilde{d} + 1 - \tilde{u}_k^2 + \mathcal{O}(\nu) \\ n < k: &\qquad 0 = 1 - \tilde{u}_n^2 + \mathcal{O}(\nu) \end{aligned} \tag{3.9}$$

where we used (3.8) to simplify the equation with $n = k$ using the connection to the element at index $n = k + 1$. We now see that at the index $n = k$, a fold bifurcation takes place at $(\tilde{u}_k, \tilde{d}, \nu) = (0, 1, 0)$. Extending this fold bifurcation to the full system on the indices $n \geq k$ and into $\nu > 0$ follows as in [3, Lemma 3.4] and is omitted. This completes the proof of the lemma. □

3.2 Next-nearest-neighbour coupling

In this subsection we provide analogous results to the nearest-neighbour bifurcation branches, but with $m = 2$, representing next-nearest-neighbour connections. The results of this subsection therefore complete the proof of Theorem 1 with $m = 2$. We will consider $N \geq 7$, since $N = 4, 5$ and $m = 2$ represent all-to-all connections, and $N = 6$ with $m = 2$ has an extra symmetry that can be exploited, as presented in Theorem 2. We present the following result, analogous to Lemma 3.1 above.

Lemma 3.3. *Fix $m = 2$ and $3 \leq k \leq \lfloor \frac{N}{2} \rfloor + 1$, then the following is true for (3.2). There are constants $d_1, \mu_1 > 0$ and a smooth function $\mu_l : [0, d_1] \to [0, \mu_1]$ such that for each $d \in (0, d_1]$ there is a pair of κ-symmetric solutions $u_l(\mu, d)$ and $v_l(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_l(d)$ and exist for all $\mu \in [\mu_l(d), \mu_1]$. These solutions are smooth in (μ, d), and for each fixed μ, we have $u_l(\mu, d) \to \bar{u}^{(k-1)}(\mu)$ and $v_l(\mu, d) \to \bar{v}^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_l(d)$ satisfies $\mu_l(d) = 3d^{\frac{2}{3}} + \mathcal{O}(d)$.*

Proof. Fix $m = 2$ and $3 \leq k \leq \lfloor \frac{N}{2} \rfloor + 1$. Then, following as in Lemma 3.1 to continue the elements with $n < k$ into $d > 0$ near $\mu = 0$ using the implicit function theorem. We again get the asymptotic expansion (3.5), and for the indices with $n \geq k$ we introduce the scaling

$$\mu = \nu^2, \qquad d = \nu^3 \tilde{d}, \qquad u_n = \nu^{\lfloor \frac{n-k}{2} \rfloor + 1} \tilde{u}_n. \tag{3.10}$$

That is, $u_k = \nu \tilde{u}_k$, $u_{k+1} = \nu \tilde{u}_{k+1}$, $u_{k+2} = \nu^2 \tilde{u}_{k+2}$, $u_{k+3} = \nu^2 \tilde{u}_{k+3}$, and so on. Putting these rescaled variables into (3.2), expanding in powers of ν, and dividing off the leading power in ν brings us to the equations, analogous to (3.7),

$$\begin{aligned} n = k: &\qquad 0 = 2\tilde{d} - \tilde{u}_k + \tilde{u}_k^3 + \mathcal{O}(\nu) \\ n = k+1: &\qquad 0 = \tilde{d} - \tilde{u}_{k+1} + \tilde{u}_{k+1}^3 + \mathcal{O}(\nu) \\ n \geq k+2: &\qquad 0 = \tilde{d}\tilde{u}_{n-2} + \tilde{d}\tilde{u}_{n-1} - \tilde{u}_n + \mathcal{O}(\nu) \end{aligned} \tag{3.11}$$

where we have used the fact that at index $n = k$ we are connected to the elements at index $k - 1, k - 2$, both of which are equal to 1 at $(\mu, d) = (0, 0)$. Similarly, since $m = 2$, the element at index $n = k + 1$ is only connected to one element (at $n = k - 1$) that is equal to 1 at $(\mu, d) = (0, 0)$. The proof is now the same as that of Lemma 3.1. □

Let us now finish the case of bifurcations near $\mu = 0$ by considering the case $k = 2$. We present the following result.

Lemma 3.4. *Fix $m = k = 2$, then the following is true for (3.2). There are constants $d_1, \mu_1 > 0$ and a smooth function $\mu_l : [0, d_1] \to [0, \mu_1]$ such that for each $d \in (0, d_1]$ there is a pair of κ-symmetric solutions $u_l(\mu, d)$ and $v_l(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_l(d)$ and exist for all $\mu \in [\mu_l(d), \mu_1]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_l(\mu, d) \to \bar{u}^{(1)}(\mu)$ and $v_l(\mu, d) \to \bar{v}^{(2)}(\mu)$ as $d \to 0^+$. The function $\mu_l(d)$ satisfies $\mu_l(d) = \frac{3}{\sqrt[3]{2}} d^{\frac{2}{3}} + \mathcal{O}(d)$.*

Proof. Following as in the proof of Lemma 3.3 up to (3.11), we now arrive at the equations

$$\begin{aligned} n = 2: &\qquad 0 = \tilde{d} - \tilde{u}_2 + \tilde{u}_2^3 + \nu(\tilde{u}_3 - 3\tilde{u}_2) + \mathcal{O}(\nu^2) \\ n = 3: &\qquad 0 = \tilde{d} - \tilde{u}_3 + \tilde{u}_3^3 + \nu(\tilde{u}_2 - 4\tilde{u}_3) + \mathcal{O}(\nu^2) \\ n \geq 4: &\qquad 0 = \tilde{d}\tilde{u}_{n-2} + \tilde{d}\tilde{u}_{n-1} - \tilde{u}_n + \mathcal{O}(\nu) \end{aligned} \tag{3.12}$$

where now, since $m = 2$, we have that the elements at indices $n = 2, 3$ are both connected to the single element at $n = 1$ that equals 1 at $(\mu, d) = (0, 0)$. Note that the equations at indices $n = 2, 3$ agree at the lowest order in ν, but differ at $\mathcal{O}(\nu)$. This comes from the symmetry imposed by our assumption that the

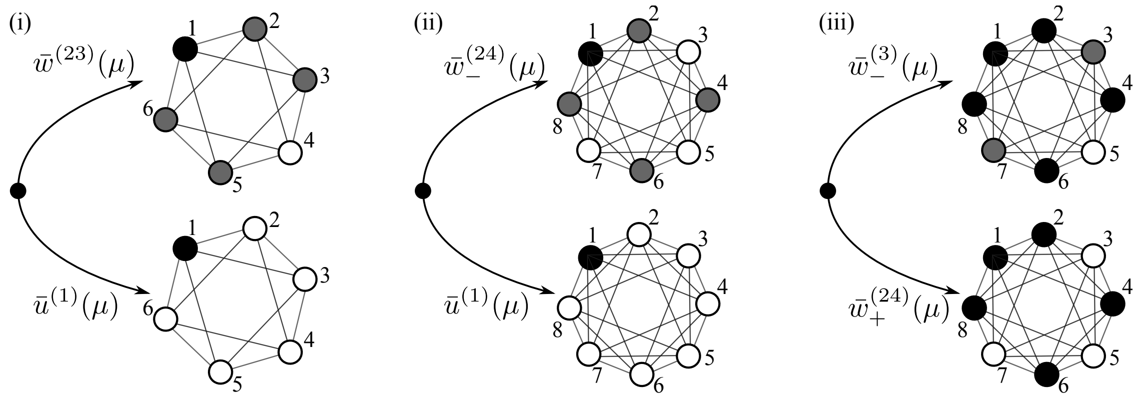

Figure 5: *Atypical bifurcations near $= 0$ for $N = 6; 8$ and $m = bN=2c$ 1. Panel (i) contains an illustration of the proof of Lemma 3.6. Panels (ii) and (iii) illustrate the bifurcations near $= 0$ when $N = 8$, as presented in Lemmas 3.8 and 3.10. In all images black circles represent elements continued from $u_+(\ )$ into $d > 0$, shaded grey circles are continued from $u_-(\ )$, and white circles are continued from 0. Connections are represented by lines connecting the circles.*

solution is invariant with respect to the action of κ, which enforces that $u_N = u_2$ and thus eliminating one of the connections at $n = 2$. In contrast, we have exactly four self-interactions at index $n = 3$ in (3.12) at order ν since the element at index $n = 3$ has no neighbours that have a symmetric restriction imposed them. From here the proof now follows as in [3, Lemma 3.3]. □

We now turn to the continuation when $\mu = 1$, which will similarly follow the proof of Lemma 3.2.

Lemma 3.5. *Fix $m = 2$ and $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$, then the following is true for (3.2). There exist constants $d_2, \mu_2 > 0$ and a smooth function $\mu_r : [0, d_2] \to [\mu_2, 1]$ such that each xed $d \in (0, d_2]$ there is a pair of κ-symmetric solutions $u_r(\mu, d)$ and $v_r(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_r(d)$ and exist for all $\mu \in [\mu_2, \mu_r(d)]$. These solutions are smooth in (μ, d), and for each xed μ we have $u_r(\mu, d) \to \bar{u}^{(k)}(\mu)$ and $v_r(\mu, d) \to \bar{v}^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_r(d)$ is given by $\mu_r(d) = 1 - 2d + \mathcal{O}(d^{\frac{3}{2}})$.*

Proof. This proof is almost the same as that of Lemma 3.2 with (3.9) replaced by

$$\begin{aligned} n = k: &\qquad 0 = -2\tilde{d} + 1 - \tilde{u}_k^2 + \mathcal{O}(\nu) \\ n = k-1: &\qquad 0 = -\tilde{d} + 1 - \tilde{u}_{k-1}^2 + \mathcal{O}(\nu) \\ n \leq k-2: &\qquad 0 = 1 - \tilde{u}_n^2 + \mathcal{O}(\nu). \end{aligned} \tag{3.13}$$

The subsequent analysis proceeds in the the same way as Lemma 3.2. □

3.3 Almost all-to-all coupling

Let us begin with the proof of Theorem 2, which features $(N, m) = (6, 2)$. With the exception of the connections involving curves continued from $\bar{w}^{(23)}(\mu)$, the analysis is the same as in the proof of Theorem 1. Hence, we will focus exclusively on the connections involving $\bar{w}^{(23)}(\mu)$. There is only one such connection near $\mu = 1$, which is illustrated for small $d > 0$ in Figure 5(i). We prove this with the following lemma.

Lemma 3.6. *Fix $(N, m) = (6, 2)$, then the following is true for (3.2). There are constants $d_1, \mu_1 > 0$ and a smooth function $\mu_l : [0, d_1] \to [0, \mu_1]$ such that for each $d \in (0, d_1]$, there is a pair of κ-symmetric solutions $u_l(\mu, d)$ and $v_l(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_l(d)$ and exist for all $\mu \in [\mu_l(d), \mu_1]$. These solutions are smooth in (μ, d), and for each xed μ we have $u_l(\mu, d) \to \bar{u}^{(1)}(\mu)$ and $v_l(\mu, d) \to \bar{w}^{(23)}(\mu)$ as $d \to 0^+$. The function $\mu_l(d)$ satis es $\mu_l(d) = \frac{3}{\sqrt[3]{2}} d^{\frac{2}{3}} + \mathcal{O}(d)$.*

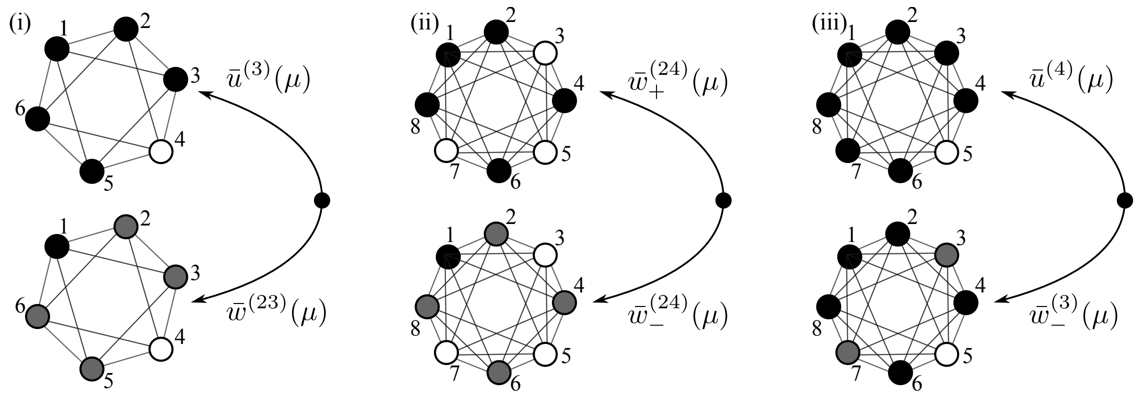

Figure 6: *Atypical bifurcations near* $= 1$ *for* $N = 6;8$ *and* $m = bN=2c$ 1. *Panel (i) contains an illustration of the proof of Lemma 3.7. Panels (ii) and (iii) illustrate the bifurcations near* $= 1$ *when* $N = 8$*, as presented in Lemmas 3.9 and 3.11. Shading and connections are the same as in Figure 5.*

Proof. This proof is handled in exactly the same way as the other proofs for bifurcations near $\mu = 0$, and so we will only seek to highlight why we have a different connection when $N = 6$ and $m = 2$. Using the implicit function theorem, we can solve (3.2) at the index $n = 1$ in a neighbourhood of $(d, \mu) = (0, 0)$ to get

$$u_1 = u_1(u^c, \mu, d) = 1 + \mathcal{O}(|\mu| + |d|\|u^c\|),$$

where $u^c = (u_2, u_3, u_4)$ since we have imposed $u_5 = u_3$ and $u_6 = u_2$ by symmetry. It then remains to solve (3.2) for the indices associated with u^c, i.e. $n = 2, 3, 4$. Specifically, we are required to solve

$$\begin{aligned} d(u_1 + u_3 + u_4 - 3u_2) + f(u_2, \mu) &= 0, \\ d(u_1 + u_2 + u_4 - 3u_3) + f(u_3, \mu) &= 0, \\ 2d(u_2 + u_3 - 2u_4) + f(u_4, \mu) &= 0, \end{aligned} \tag{3.14}$$

where we have simplified the equations using the symmetric restrictions $u_5 = u_3$ and $u_6 = u_2$. One can see that the resulting equations are invariant with respect to the action $(u_2, u_3) \mapsto (u_3, u_2)$, and therefore we may restrict ourselves to the symmetric subspace that has $u_2 = u_3$. Upon imposing this restriction we may then follow as in the previous proofs to obtain the desired result. □

Near $\mu = 1$ we find that the branch continued from $\bar{w}^{(23)}(\mu)$ connects to $\bar{u}^{(3)}(\mu)$, summarized in the following lemma and illustrated in Figure 6(i). The lemma is stated without proof since it is identical to the previous lemmas.

Lemma 3.7. *Fix $(N, m) = (6, 2)$, then the following is true for (3.2). There exist constants $d_2, \mu_2 > 0$ and a smooth function $\mu_r : [0, d_2] \to [\mu_2, 1]$ such that for each xed $d \in (0, d_2]$, there is a pair of κ-symmetric solutions $u_r(\mu, d)$ and $v_r(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_r(d)$ and exist for all $\mu \in [\mu_2, \mu_r(d)]$. These solutions are smooth in (μ, d), and for each xed μ we have $u_r(\mu, d) \to \bar{u}^{(3)}(\mu)$ and $v_r(\mu, d) \to \bar{w}^{(23)}(\mu)$ as $d \to 0^+$. The function $\mu_r(d)$ is given by $\mu_r(d) = 1 - 2d + \mathcal{O}(d^{\frac{3}{2}})$.*

Turning now to the case of $(N, m) = (8, 3)$, we remark that much of the analysis is similar. Recall that the set $\Gamma_{8,3}$ is characterized by following along solution branches at $d = 0$ given by the following sequence:

$$\bar{v}^{(1)}(\mu) \to \bar{u}^{(1)}(\mu) \to \bar{w}_{-}^{(24)}(\mu) \to \bar{w}_{+}^{(24)}(\mu) \to \bar{w}_{-}^{(3)}(\mu) \to \bar{u}^{(4)}(\mu) \to \bar{v}^{(4)}(\mu).$$

The connections between $\bar{u}$ and $\bar{v}$ elements are handled as in the proof of Theorem 1, while the connections between any of the $\bar{w}$ elements are summarized in the following lemmas. We refer the reader to Figure 5 for illustrations of the connections near $\mu = 0$ and to Figure 6 for illustrations near $\mu = 1$. The lemmas are listed according to the order of the sequence above, while the proofs are omitted since they are similar to much of the work performed in this section. Together these results complete the proof of Theorem 3.

Lemma 3.8. *Fix $(N, m) = (8, 3)$, then the following is true for (3.2). There are constants $d_1, \mu_1 > 0$ and a smooth function $\mu_l : [0, d_1] \to [0, \mu_1]$ such that for each $d \in (0, d_1]$ there is a pair of κ-symmetric solutions $u_l(\mu, d)$ and $v_l(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_l(d)$ and exist for all $\mu \in [\mu_l(d), \mu_1]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_l(\mu, d) \to \bar{u}^{(1)}(\mu)$ and $v_l(\mu, d) \to \bar{w}_-^{(24)}(\mu)$ as $d \to 0^+$. The function $\mu_l(d)$ satisfies $\mu_l(d) = \frac{3}{\sqrt[3]{2}} d^{\frac{2}{3}} + \mathcal{O}(d)$.*

Lemma 3.9. *Fix $(N, m) = (8, 3)$, then the following is true for (3.2). There exist constants $d_2, \mu_2 > 0$ and a smooth function $\mu_r : [0, d_2] \to [\mu_2, 1]$ such that for each fixed $d \in (0, d_2]$ there is a pair of κ-symmetric solutions $u_r(\mu, d)$ and $v_r(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_r(d)$ and exist for all $\mu \in [\mu_2, \mu_r(d)]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_r(\mu, d) \to \bar{w}_-^{(24)}(\mu)$ and $v_r(\mu, d) \to \bar{w}_+^{(24)}(\mu)$ as $d \to 0^+$. The function $\mu_r(d)$ is given by $\mu_r(d) = 1 - 2d + \mathcal{O}(d^{\frac{3}{2}})$.*

Lemma 3.10. *Fix $(N, m) = (8, 3)$, then the following is true for (3.2). There are constants $d_1, \mu_1 > 0$ and a smooth function $\mu_l : [0, d_1] \to [0, \mu_1]$ such that for each $d \in (0, d_1]$ there is a pair of κ-symmetric solutions $u_l(\mu, d)$ and $v_l(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_l(d)$ and exist for all $\mu \in [\mu_l(d), \mu_1]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_l(\mu, d) \to \bar{w}_+^{(23)}(\mu)$ and $v_l(\mu, d) \to \bar{w}_-^{(3)}(\mu)$ as $d \to 0^+$. The function $\mu_l(d)$ satisfies $\mu_l(d) = \frac{3}{\sqrt[3]{2}} d^{\frac{2}{3}} + \mathcal{O}(d)$.*

Lemma 3.11. *Fix $(N, m) = (8, 3)$, then the following is true for (3.2). There exist constants $d_2, \mu_2 > 0$ and a smooth function $\mu_r : [0, d_2] \to [\mu_2, 1]$ such that for each fixed $d \in (0, d_2]$ there is a pair of κ-symmetric solutions $u_r(\mu, d)$ and $v_r(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_r(d)$ and exist for all $\mu \in [\mu_2, \mu_r(d)]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_r(\mu, d) \to \bar{w}_-^{(3)}(\mu)$ and $v_r(\mu, d) \to \bar{u}^{(4)}(\mu)$ as $d \to 0^+$. The function $\mu_r(d)$ is given by $\mu_r(d) = 1 - 2d + \mathcal{O}(d^{\frac{3}{2}})$.*

3.4 All-to-all coupling

Let us now consider the case of all-to-all coupling. That is, $m = \lfloor \frac{N}{2} \rfloor$, and so every element is connected to all other elements. Then, our interest in G_k-invariant solutions means that the system (2.16) reduces to solving the two equations

$$\begin{cases} d(N-k)(v_2 - v_1) + f(v_1, \mu) = 0, & (3.15a) \\ dk(v_1 - v_2) + f(v_2, \mu) = 0 & (3.15b) \end{cases}$$

where v_1 denotes the values of the vector in $\mathbb{R}^N$ at the first k indices and v_2 the values of the last $N - k$. As stated at the beginning of this section, we clearly have that for μ belonging to any compact interval of $[0, 1]$ we can continue a solution of (3.1) at $d = 0$ with $u_n \in \{0, u_\pm(\mu)\}$ for all $n = 2, \ldots, N$ regularly into $d > 0$ with the implicit function theorem. Moreover, from the form of (3.15), we may restrict $\mathcal{F}$ to a G_k-invariant subspace to guarantee that solutions which are G_k-invariant at $d = 0$ persist into $d > 0$ with the same symmetry. Hence, to prove Theorem 4 we again need only check continuation into $d > 0$ of the connections between the elements $\bar{a}_\pm^{(k)}(\mu)$, $\bar{b}^{(k)}(\mu)$, $\bar{c}_\pm^{(k)}(\mu)$, and $\bar{d}^{(k)}(\mu)$ near $\mu = 0, 1$.

We begin with the following lemma which details the continuation into $d > 0$ of the connections between $\bar{a}_+^{(k)}(\mu)$ and $\bar{b}^{(k)}(\mu)$ and $\bar{c}_+^{(k)}(\mu)$ and $\bar{d}^{(k)}(\mu)$, near $\mu = 0$.

Lemma 3.12. *For any $N \geq 2$ and $k = 1, \ldots, N - 1$, the following is true for (3.2).*

1. *There are constants $d_{ab}, \mu_{ab} > 0$ and a smooth function $\mu_l : [0, d_{ab}] \to [0, \mu_{ab}]$ such that for each $d \in (0, d_{ab}]$ there is a pair of G_k-symmetric solutions $u_{ab}(\mu, d)$ and $v_{ab}(\mu, d)$ of (3.2) that bifurcate at*

a fold bifurcation at $\mu = \mu_{ab}(d)$ *and exist for all* $\mu \in [\mu_{ab}(d), \mu_1]$*. These solutions are smooth in* (μ, d)*, and for each xed* μ *we have* $u_{ab}(\mu, d) \to \bar{a}_+^{(k)}(\mu)$ *and* $v_{ab}(\mu, d) \to \bar{b}^{(k)}(\mu)$ *as* $d \to 0^+$*. The function* $\mu_{ab}(d)$ *satis es* $\mu_{ab}(d) = 3\sqrt[3]{\frac{k^2}{4}}d^{\frac{2}{3}} + \mathcal{O}(d)$*.*

2. *There are constants* $d_{cd}, \mu_{cd} > 0$ *and a smooth function* $\mu_l : [0, d_{cd}] \to [0, \mu_{cd}]$ *such that for each* $d \in (0, d_{cd}]$ *there is a pair of* G_k*-symmetric solutions* $u_{cd}(\mu, d)$ *and* $v_{cd}(\mu, d)$ *of* (3.2) *that bifurcate at a fold bifurcation at* $\mu = \mu_{cd}(d)$ *and exist for all* $\mu \in [\mu_{ab}(d), \mu_1]$*. These solutions are smooth in* (μ, d)*, and for each xed* μ *we have* $u_{cd}(\mu, d) \to \bar{c}_+^{(k)}(\mu)$ *and* $v_{cd}(\mu, d) \to \bar{d}^{(k)}(\mu)$ *as* $d \to 0^+$*. The function* $\mu_{cd}(d)$ *satis es* $\mu_{cd}(d) = 3\sqrt[3]{\frac{(N-k)^2}{4}}d^{\frac{2}{3}} + \mathcal{O}(d)$*.*

Proof. We will only prove the first case in the lemma since the second can simply be obtained by exchanging v_1 and v_2 in (3.15) in what follows. In this notation the functions $\bar{a}_\pm^{(k)}(\mu)$ and $\bar{b}^{(k)}(\mu)$ correspond to solutions of (3.15) with $d = 0$ as well as $(v_1, v_2) = (u_\pm(\mu), 0)$ and $(u_+(\mu), u_-(\mu))$, respectively, for all $\mu \in [0, 1]$. For convenience, we will denote

$$\begin{aligned} \mathcal{F}_1(v_1, v_2, \mu, d) &= d(N-k)(v_2 - v_1) + f(v_1, \mu) \\ \mathcal{F}_2(v_1, v_2, \mu, d) &= dk(v_1 - v_2) + f(v_2, \mu) \end{aligned} \tag{3.16}$$

to emphasize the connection with system (3.15) with (3.2) and the function $\mathcal{F}$, defined in (3.1). Importantly, solutions of $\mathcal{F}_1 = \mathcal{F}_2 = 0$ represent G_k-invariant steady-state solutions of (3.1).

In the present scenario, we recall that $u_+(0) = 1$ and restrict ourselves to a neighbourhood of the solution $(v_1, v_2, \mu, d) = (1, 0, 0, 0)$ to (3.15). From the previous discussion, this represents a G_k-invariant neighbourhood of the solution $(u, \mu, d) = (\bar{a}_+(0), 0, 0)$ to $\mathcal{F}$ in (3.1). Then, since

$$\frac{\partial \mathcal{F}_1}{\partial v_1}(1, 0, 0, 0) = f_u(1, 0) \neq 0, \tag{3.17}$$

the implicit function theorem guarantees the existence of a unique smooth function $v_1^*(v_2, \mu, d)$ satisfying $v_1^*(0, 0, 0) = 1$ and $\mathcal{F}_1(v_1^*(v2, \mu, d), v_2, \mu, d) = 0$ for all (v_2, μ, d) in a neighbourhood of $(0, 0, 0)$. Hence, the Taylor series of $v_1^*(v_2, \mu, d)$ about $(v_2, \mu, d) = (0, 0, 0)$ is given by

$$v_1^*(v_2, \mu, d) = 1 + \mathcal{O}(|\mu| + |d| + |v_2||d|). \tag{3.18}$$

The analysis now reduces to solving $\mathcal{F}_2(v_1, v_2, \mu, d) = 0$ with $v_1 = v_1^*(v_2\mu, d)$ in a neighbourhood of $(v_2, \mu, d) = (0, 0, 0)$.

As in the previous lemmas of this section, let us introduce the re-scaled variables

$$\mu = \nu^2, \qquad d = \nu^3\tilde{d}, \qquad v_2 = \nu\tilde{v}_2, \tag{3.19}$$

for $|\nu| \ll 1$, so that (3.18) becomes

$$v_1^*(v_2, \mu, d) = 1 + \mathcal{O}(\nu^2). \tag{3.20}$$

Then, putting $v_1 = v_1^*(v_2\mu, d)$ into $\mathcal{F}_2$ and expanding using (3.18) and (3.3) gives

$$\mathcal{F}_2(v_1^*(v_2, \mu, d), v_2, \mu, d) = \nu^3(k\tilde{d} - \tilde{v}_2 + \tilde{v}_2^3) + \mathcal{O}(\nu^4). \tag{3.21}$$

Hence, solving $\mathcal{F}_2(v_1^*(v_2, \mu, d), v_2, \mu, d) = 0$ is equivalent to solving

$$0 = k\tilde{d} - \tilde{v}_2 + \tilde{v}_2^3 + \mathcal{O}(\nu) \tag{3.22}$$

after dividing off ν^3. It is therefore clear that the above expression experiences a fold bifurcation at

$$(\tilde{v}_2, \tilde{d}, \nu) = \left(\sqrt[3]{\frac{k}{2}}, \frac{2}{3\sqrt{3}k}, 0\right), \tag{3.23}$$

for which we may follow as in the proofs of the previous lemmas to demonstrate the persistence of this fold into sufficiently small $\nu > 0$. This completes the proof. □

Let us now turn to the bifurcation near $\mu = 1$. The following lemma details the persistence of the connections between $\bar{a}_-^{(k)}(\mu)$ and $\bar{a}_+^{(k)}(\mu)$ and $\bar{c}_-^{(k)}(\mu)$ and $\bar{c}_+^{(k)}(\mu)$ into $d > 0$.

Lemma 3.13. *For any $N \geq 2$ and $k = 1, \ldots, N-1$, the following is true for (3.2).*

1. *There exist constants $d_{aa}, \mu_{aa} > 0$ and a smooth function $\mu_{aa} : [0, d_{aa}] \to [\mu_{aa}, 1]$ such that for each fixed $d \in (0, d_{aa}]$ there is a pair of G_k-symmetric solutions $u_{aa}(\mu, d)$ and $v_{aa}(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_{aa}(d)$ and exist for all $\mu \in [\mu_{aa}, \mu_{aa}(d)]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_{aa}(\mu, d) \to \bar{a}_-^{(k)}(\mu)$ and $v_{aa}(\mu, d) \to \bar{a}_+^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_{aa}(d)$ satisfies $\mu_{aa}(d) = 1 - (N-k)d + \mathcal{O}(d^{\frac{3}{2}})$.*
2. *There exist constants $d_{cc}, \mu_{cc} > 0$ and a smooth function $\mu_{cc} : [0, d_{cc}] \to [\mu_{cc}, 1]$ such that for each fixed $d \in (0, d_{cc}]$ there is a pair of G_k-symmetric solutions $u_{cc}(\mu, d)$ and $v_{cc}(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_{cc}(d)$ and exist for all $\mu \in [\mu_{cc}, \mu_{cc}(d)]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_{cc}(\mu, d) \to \bar{c}_-^{(k)}(\mu)$ and $v_{cc}(\mu, d) \to \bar{c}_+^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_{cc}(d)$ satisfies $\mu_{cc}(d) = 1 - kd + \mathcal{O}(d^{\frac{3}{2}})$.*

Proof. Since we are interested in G_k-invariant solutions of (3.2), we may again proceed as in the proof of Lemma 3.12 and reduce to solving the equations (3.15). As in Lemma 3.12, we will only prove the first statement since the second follows from simply interchanging v_1 and v_2 in what follows. The difference is that now we focus a neighbourhood of the solution $(v_1, v_2, \mu, d) = (1, 0, 1, 0)$, corresponding to a G_k-invariant neighbourhood of the solution $(u, \mu, d) = (a_\pm^{(k)}(1), 1, 0)$ of (3.2). Since we have

$$\frac{\partial \mathcal{F}_2}{\partial v_2}(1, 0, 1, 0) = f_u(0, 1) \neq 0, \tag{3.24}$$

we may apply the implicit function theorem to obtain a smooth function $v_2^*(v_1, \mu, d)$ defined in a neighbourhood of $(v_1, \mu, d) = (1, 1, 0)$ satisfying $v_2^*(1, 1, 0) = 0$ and $\mathcal{F}_2(v_1, v_2^*(v_1, \mu, d), \mu, d) = 0$. Solving

$$\mathcal{F}_1(v_1, v_2^*(v_1, \mu, d), \mu, d) = 0 \tag{3.25}$$

in a neighbourhood of $(v_1, \mu, d) = (1, 1, 0)$ then proceeds as in the lemmas of the previous subsections and is therefore omitted. □

Next, the following lemma details the continuation into $d > 0$ of the connection between $\bar{b}^{(k)}(\mu)$ and $\bar{d}^{(k)}(\mu)$ near $\mu = 1$. We comment that this lemma covers the connection in the top right corner of panel (iii) in Figure 2.

Lemma 3.14. *For any $N \geq 2$ and $k = 1, \ldots, N-1$, the following is true for (3.2). There exist constants $d_{bd}, \mu_{bd} > 0$ and a smooth function $\mu_{bd} : [0, d_{bd}] \to [\mu_{bd}, 1]$ such that for each fixed $d \in (0, d_{bd}]$ there is a pair of G_k-symmetric solutions $u_{bd}(\mu, d)$ and $v_{bd}(\mu, d)$ of (3.2) that bifurcate at a fold bifurcation at $\mu = \mu_{aa}(d)$ and exist for all $\mu \in [\mu_{bd}, \mu_{bd}(d)]$. These solutions are smooth in (μ, d), and for each fixed μ we have $u_{bd}(\mu, d) \to \bar{b}^{(k)}(\mu)$ and $v_{bd}(\mu, d) \to \bar{d}^{(k)}(\mu)$ as $d \to 0^+$. The function $\mu_{bd}(d)$ satisfies $\mu_{bd}(d) = 1 - k(N-k)d^2 + \mathcal{O}(d^3)$.*

Proof. We consider the neighbourhood of the solution $(v_1, v_2, \mu, d) = (1, 1, 1, 0)$ to (3.15). For simplicity, denote $\tilde{\mu} = 1 - \mu$, $\tilde{v}_1 = v_1 - 1$ and $\tilde{v}_2 = v_2 - 1$. Using the expansion (3.4) these new variables transform (3.15) to

$$\begin{cases} \tilde{\mathcal{F}}_1(\tilde{v}_1, \tilde{v}_2, \tilde{\mu}, d) = d(N-k)(\tilde{v}_2 - \tilde{v}_1) + \tilde{\mu} - \tilde{v}_1^2 - b\tilde{\mu}\tilde{v}_1 + \mathcal{O}(\tilde{\mu}^2 + |\tilde{v}|^4) = 0, & (3.26\text{a}) \\ \tilde{\mathcal{F}}_2(\tilde{v}_1, \tilde{v}_2, \tilde{\mu}, d) = dk(\tilde{v}_1 - \tilde{v}_2) + \tilde{\mu} - \tilde{v}_2^2 - b\tilde{\mu}\tilde{v}_2 + \mathcal{O}(\tilde{\mu}^2 + |\tilde{v}|^4) = 0 & (3.26\text{b}) \end{cases}$$

where $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$. We can apply the implicit function theorem to equation (3.26a) and find a unique smooth function $\tilde{\mu}^*(\tilde{v}_1, \tilde{v}_2, d)$, defined in a neighbourhood of $(\tilde{v}_1, \tilde{v}_2, d) = (0, 0, 0)$, satisfying $\tilde{\mu}^*(0, 0, 0) = 0$ and $\tilde{\mathcal{F}}_1(\tilde{v}_1, \tilde{v}_2, \tilde{\mu}^*(\tilde{v}_1, \tilde{v}_2, d), d) = 0$, in the neighbourhood of $(0, 0, 0, 0)$. Moreover, we have the expansion

$$\tilde{\mu}^*(\tilde{v}_1, \tilde{v}_2, d) = -d(N-k)(\tilde{v}_2 - \tilde{v}_1) + \tilde{v}_1^2 + \mathcal{O}(|d||\tilde{v}|^2 + |\tilde{v}|^4) \tag{3.27}$$

Next, we subtract the two equations in (3.15) to obtain

$$dN(v_2 - v_1) - f(v_1, \mu) - f(v_2, \mu) = 0. \tag{3.28}$$

Defining

$$g(\tilde{v}_1, \tilde{v}_2, \tilde{\mu}) := \int_0^1 f_u(v_2 + \tau(v_1 - v_2), \mu) d\tau = -\tilde{v}_1 - \tilde{v}_2 - \tilde{\mu} + \mathcal{O}(\tilde{\mu}^2 + |\tilde{v}|^3),$$

we can write (3.28) in new variables as

$$dN(\tilde{v}_2 - \tilde{v}_1) + f(\tilde{v}_1, \tilde{\mu}) - f(\tilde{v}_2, \tilde{\mu}) = dN(\tilde{v}_2 - \tilde{v}_1) - (\tilde{v}_2 - \tilde{v}_1) g(\tilde{v}_1, \tilde{v}_2, \tilde{\mu}) = 0.$$

Assuming $\tilde{v}_1 \neq \tilde{v}_2$, we can divide by $\tilde{v}_2 - \tilde{v}_1$ to arrive at

$$-dN + g(\tilde{v}_1, \tilde{v}_2, \tilde{\mu}) = -dN + \tilde{v}_1 + \tilde{v}_2 + \tilde{\mu} + \mathcal{O}(\tilde{\mu}^2 + |\tilde{v}|^3) = 0.$$

We can further plug in the expression for $\tilde{\mu}$ in equation (3.27) and find

$$-dN + \tilde{v}_1 + \tilde{v}_2 - d(N-k)(\tilde{v}_2 - \tilde{v}_1) + \tilde{v}_1^2 + \mathcal{O}(d + |\tilde{v}|)|\tilde{v}|^2 = 0.$$

Using the implicit function theorem to solve for $\tilde{v}_2$ as a function of $(\tilde{v}_1, d)$ in a neighbourhood of $(0, 0)$ gives

$$\tilde{v}_2^*(\tilde{v}_1, d) = -Nd - \tilde{v}_1 + \mathcal{O}(\tilde{v}_1^2 + d^2 + d\tilde{v}_1), \tag{3.29}$$

and putting this into (3.27), we finally find

$$\tilde{\mu}^*(\tilde{v}_1, d) = \tilde{v}_1^2 + 2(N-k)\tilde{v}_1 d + N(N-k)d^2 + \mathcal{O}(\tilde{v}_1^3 + d^3). \tag{3.30}$$

Hence, a fold bifurcation takes place at $\tilde{v}_1 = \tilde{v}_1^*(d) = -(N-k)d + \mathcal{O}(d^2)$ when d is near 0. Putting $\tilde{v}_1^*(d)$ into $\tilde{\mu}^*(\tilde{v}_1, d)$, as defined above, results in the expansion given in the statement of the lemma for the location of the fold bifurcation. This completes the proof. □

Finally, we prove that the branch of patterns disappears near $\mu = 0$ for $0 < d \ll 1$ through a collision with the branch of homogeneous patterns at which all nodes are equal to the middle root $u_-(\mu)$ of the nonlinearity $f(u, \mu)$. Based on the work of the previous lemmas, we opt to state this lemma for the system (3.15) for the convenience of the reader.

Lemma 3.15. *For any $N \geq 2$ and $k = 1, \ldots, N-1$, equation (3.15) admits a two-parameter family $(v_1^*, v_2^*, d_*, \mu_*)(s, a)$ of solutions that are de ned for $0 \leq s \ll 1$ and $1 \leq a \leq 2$ where $(v_1^*, v_2^*) = (s, as)$ and*

$$d_*(s, a) = \frac{a(1+a)}{N + k(1-a)} s^2 + a\mathcal{O}(s^4), \qquad \mu_*(s, a) = \left[1 + \frac{ka(1-a^2)}{N + k(1-a)}\right] s^2 + \mathcal{O}(s^4).$$

These solutions terminate onto the homogeneous solution branch $v_1 = v_2 = u_-(\mu_*(s, 1))$ when $a = 1$. Furthermore, the intersection of this two-parameter family with the hyperplane $d = \epsilon^2$ consists of a curve obtained by setting $s = s_*(a)\epsilon + \mathcal{O}(\epsilon^2)$ parametrized by $0 < a_0 \leq a \leq 2$ for $0 < a_0 < 1$ xed and $0 < \epsilon \ll 1$.

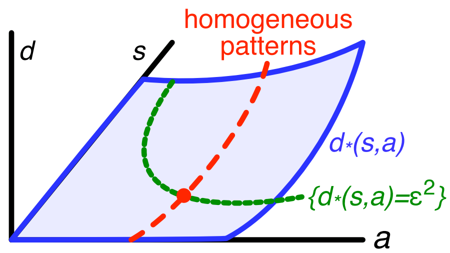

Figure 7: *Shown is the bifurcation diagram near $\;= 0$ for all-to-all coupling.*

Proof. Recall that we need to solve the system (3.15) given by

$$\begin{cases} d(N-k)(v_2 - v_1) + f(v_1, \mu) = 0, & (3.31a) \\ dk(v_1 - v_2) + f(v_2, \mu) = 0 & (3.31b) \end{cases}$$

where v_1 denotes the values of the vector in R^N at the first k indices and v_2 the values of the last $N-k$. Recall also that

$$f(u, \mu) = -\mu u + u^3 + \mathcal{O}(\mu^2 u + \mu u^3 + u^5)$$

First, when $u = v_1 = v_2$, equation (3.31) reduces to $f(u, \mu) = 0$ which admits the solution branch $(u, \mu) = (u_-(s), \mu_h(s))$ for $0 \le s \ll 1$ with $u_-(0) = \mu_h(0) = 0$ and $\mu_h(s) > 0$ for $s > 0$. Next, we subtract the two equations in (3.31) to obtain

$$dN(v_2 - v_1) + f(v_1, \mu) - f(v_2, \mu) = 0. \tag{3.32}$$

Defining $v = (v_1, v_2)$ and

$$g(v_1, v_2, \mu) := \int_0^1 f_u(v_2 + \tau(v_1 - v_2), \mu) d\tau = -\mu + v_1^2 + v_1 v_2 + v_2^2 + \mathcal{O}(\mu^2 + |v|^4),$$

we can write (3.32) as

$$dN(v_2 - v_1) + f(v_1, \mu) - f(v_2, \mu) = dN(v_2 - v_1) + (v_1 - v_2) g(v_1, v_2, \mu) = 0.$$

Since we already analysed the case $v_1 = v_2$, we can divide by $v_1 - v_2$ to arrive at

$$-dN + g(v_1, v_2, \mu) = -dN - \mu + v_1^2 + v_1 v_2 + v_2^2 + \mathcal{O}(\mu^2 + |v|^4) = 0,$$

which we can solve for μ as a function $\mu_*(v_1, v_2, d)$ near $(v_1, v_2, d) = 0$ where

$$\mu_*(v_1, v_2, d) = v_1^2 + v_1 v_2 + v_2^2 - dN + \mathcal{O}(d^2 + |v|^4).$$

It therefore suffices to solve (3.31b) which becomes

$$\begin{aligned} 0 &= dk(v_1 - v_2) + f(v_2, \mu) \\ &= dk(v_1 - v_2) - \mu_*(v_1, v_2, d) v_2 + v_2^3 + v_2 \mathcal{O}(\mu_*(v_1, v_2, d)^2 + \mu_*(v_1, v_2, d) v_2^2 + v_2^4) \\ &= dk(v_1 - v_2) + v_2(dN - v_1^2 - v_1 v_2 + \mathcal{O}(d^2 + |v|^4)) \end{aligned}$$

and finally at

$$dkv_1 + d(N-k)v_2 - v_1 v_2 (v_1 + v_2) + v_2 \mathcal{O}(d^2 + |v|^4) = 0. \tag{3.33}$$

To solve (3.31b), we set $v = (v_1, v_2) = (1, a)s$ with $1 \le a \le 2$ so that (3.33) becomes

$$dks + d(N-k)as - a(1+a)s^3 + as\mathcal{O}(d^2 + s^4) = 0.$$

Dividing by the trivial solution $s = 0$, we obtain

$$d(N + k(1 - a)) - a(1 + a)s^2 + a\mathcal{O}(d^2 + s^4) = 0, \tag{3.34}$$

which we can solve for $d = d_*(s, a)$ for $0 \leq s \ll 1$ and $1 \leq a \leq 2$ near $(u, d) = 0$.

In summary, (3.31) admits a two-parameter family $(d_*, \mu_*)(s, a)$ and $(v_1, v_2) = (1, a)s$ of solutions that are defined for $0 \leq s \ll 1$ and $1 \leq a \leq 2$ where

$$d_*(s, a) = \frac{a(1 + a)}{N + k(1 - a)} s^2 + a\mathcal{O}(s^4), \qquad \mu_*(s, a) = \left[1 + \frac{ka(1 - a^2)}{N + k(1 - a)}\right] s^2 + \mathcal{O}(s^4).$$

Note that upon setting $a = 1$ so that $v_1 = v_2$ the resulting one-parameter branch becomes part of the homogeneous solution family $v_1 = v_2 = u_-(\mu_*(s, 1))$ that we discussed above. In particular, the nonhomogeneous branches for $a \neq 1$ bifurcate from the homogeneous solutions that have $a = 1$.

Finally, we analyse the intersection of the two-parameter family $(v_1, v_2, \mu_*, d_*)(s, a)$ with the hyperplane $d = \epsilon^2$ for $0 < \epsilon \ll 1$. Setting $d = \epsilon^2$ and $s = b\epsilon$, equation (3.34) becomes

$$N + k(1 - a) - a(1 + a)b^2 + a\mathcal{O}(\epsilon^2) = 0$$

after dividing by ϵ^2. For each $0 < a_0 \ll 1$, this equation can be solved for

$$b = b_*(a, \epsilon) = \sqrt{\frac{N + k(1 - a)}{a(1 + a)}} + \mathcal{O}(\epsilon^2)$$

where $a_0 \leq a \leq 2$ and $0 < \epsilon \ll 1$. This proves that the solution branches are indeed curves that emerge from the homogeneous solutions for each $0 < d \ll 1$ fixed. □

4 Discussion

In this paper, we characterized the branches of localized steady states of lattice dynamical systems on a ring with N nodes with sparse symmetric coupling and all-to-all coupling. We found snaking branches with $N/2$ saddle-nodes on each side for sparse coupling with interaction range $m = 1, 2$ and a figure-eight branch for all-to-all coupling. We also showed that the solution branch for all-to-all coupling terminates at the branch of homogeneous patterns where all nodes attain the same value on the unstable middle branch of the zero set of the nonlinearity $f(u, \mu)$.

As indicated in Figure 2, we do not know whether the branches for sparse coupling terminate at homogeneous solutions branches or connect to each other. Numerical continuation turns out to be complicated and intricate for small coupling strengths since several eigenvalues (which are seemingly not enforced by symmetry) are very close to zero, thus making it difficult to trust continuation as numerical solution curves may jump onto different branches.

We note that we focused on symmetric nearest-neighbour and next-nearest-neighbour sparse coupling. We conjecture that sparse coupling will always lead to snaking curves that exhibit $\lfloor \frac{N}{2} \rfloor$ saddle nodes as long as the interaction range m is strictly less than $\lfloor \frac{N}{2} \rfloor - 1$. Figure 8 indicates that the case of almost all-to-all coupling (when $m = \lfloor \frac{N}{2} \rfloor - 1$) is more complicated. We further expect that moving to asymmetric coupling on the ring will result in more intricate bifurcation curves that would be difficult to treat in a completely general analytical setting.